Homogenization of high-contrast two-phase conductivities perturbed by a magnetic field. Comparison between dimension two and dimension three.

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Abstract

Homogenized laws for sequences of high-contrast two-phase non-symmetric conductivities perturbed by a parameter h are derived in two and three dimensions. The parameter h characterizes the antisymmetric part of the conductivity for an idealized model of a conductor in the presence of a magnetic field. In dimension two an extension of the Dykhne transformation to non-periodic high conductivities permits to prove that the homogenized conductivity depends on h through some homogenized matrix-valued function obtained in the absence of a magnetic field. This result is improved in the periodic framework thanks to an alternative approach, and illustrated by a cross-like thin structure. Using other tools, a fiber-reinforced medium in dimension three provides a quite different homogenized conductivity.

Keywords: homogenization, high-contrast conductivity, magneto-transport, strong field, two-phase composites.

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1 Introduction

The mathematical theory of homogenization for second-order elliptic partial differential equations has been widely studied since the pioneer works of Spagnolo on G-convergence [40], of Murat, Tartar on H-convergence [37, 38], and of Bensoussan, Lions, Papanicolaou on periodic structures [2], in the framework of uniformly bounded (both from below and above) sequences of conductivity matrix-valued functions. It is also known since the end of the seventies [24, 31] (see also the extensions [1, 22, 11, 32]) that the homogenization of the sequence of conductivity problems, in a bounded open set Ω of $\mathbb{R}^3$,

$$\begin{cases} \operatorname{div}(\sigma_n \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with a uniform boundedness from below but not from above for σ_n, may induce nonlocal effects. However, the situation is radically different in dimension two since the nature of problem (1.1) is shown [10, 13] to be preserved in the homogenization process provided that the sequence σ_n is uniformly bounded from below.

H-convergence theory includes the case of non-symmetric conductivities in connection with the Hall effect [28] in electrodynamics (see, e.g., [33, 39]). Indeed, in the presence of a constant magnetic field the conductivity matrix is modified and becomes non-symmetric. Here, we consider an idealized model of an isotropic conductivity $\sigma(h)$ depending on a parameter h which characterizes the antisymmetric part of the conductivity in the following way:

- in dimension two,

$$\sigma(h) = \alpha I_2 + \beta h J, \quad J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \tag{1.2}$$

  where α, β are scalar an $h \in \mathbb{R}$,

- in dimension three,

$$\sigma(h) = \alpha I_3 + \beta \mathscr{E}(h), \quad \mathscr{E}(h) := \left(\begin{smallmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{smallmatrix}\right), \tag{1.3}$$

  where α, β are scalar and $h \in \mathbb{R}^3$.

Since the seminal work of Bergman [3] the influence of a low magnetic field in composites has been studied for two-dimensional composites [34, 4, 17], and for columnar composites [7, 5, 8, 26, 27]. The case of a strong field, namely when the symmetric part and the antisymmetric part of the conductivity are of the same order, has been also investigated [6, 9]. Moreover, dimension three may induce anomalous homogenized Hall effects [20, 18, 19] which do not appear in dimension two [17].

In the context of high-contrast problems the situation is more delicate when the conductivities are not symmetric. An extension in dimension two of H-convergence for non-symmetric and non-uniformly bounded conductivities was obtained in [14] thanks to an appropriate div-curl lemma. More recently, the Keller, Dykhne [30, 23] two-dimensional duality principle which claims that the mapping

$$A \mapsto \frac{A^{\mathrm{T}}}{\det A} \tag{1.4}$$

is stable under homogenization, was extended to high-contrast conductivities in [16]. However, the homogenization of both high-contrast and non-symmetric conductivities has not been precisely studied in the context of the strong field magneto-transport especially in dimension three. In this paper we establish an effective perturbation law for a mixture of two high-contrast isotropic phases in the presence of a magnetic field. The two-dimensional case is performed in a general way for non-periodic and periodic microstructures. It is then compared to the case of a three-dimensional fiber-reinforced microstructure.

In dimension two, following the modelization (1.2), consider a sequence $\sigma_n(h)$ of isotropic two-phase matrix-valued conductivities perturbed by a fixed constant $h \in \mathbb{R}$, and defined by

$$\sigma_n(h) := (1-\chi_n)\big(\alpha_1 I_2 + \beta_1 h\, J\big) + \chi_n\big(\alpha_{2,n} I_2 + \beta_{2,n} h\, J\big), \tag{1.5}$$

where χ_n is the characteristic function of phase 2, with volume fraction $\theta_n \to 0$, $\alpha_1 > 0$, β_1 are the constants of the low conducting phase 1, and $\alpha_{2,n} \to \infty$, $\beta_{2,n}$ are real sequences of the highly conducting phase 2 where $\beta_{2,n}$ is possibly unbounded. The coefficients α_1 and β_1, respectively $\alpha_{2,n}$ and $\beta_{2,n}$ also have the same order of magnitude according to the strong field assumption. Assuming that the sequence $\theta_n^{-1}\chi_n$ converges weakly-$*$ in the sense of the Radon measures to a bounded function, and that $\theta_n\alpha_{2,n}$, $\theta_n\beta_{2,n}$ converge respectively to constants $\alpha_2 > 0$, β_2, we prove (see Theorem 2.2) that the perturbed conductivity $\sigma_n(h)$ converges in an appropriate sense of H-convergence (see Definition 1.1) to the homogenized matrix-valued function

$$\sigma_*(h) = \sigma_*^0\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2\, h^2\big) + \beta_1 h\, J, \tag{1.6}$$

for some matrix-valued function σ_*^0 which depends uniquely on the microstructure χ_n in the absence of a magnetic field, and is defined for a subsequence of n. The proof of the result is based on a Dykhne transformation of the type

$$A_n \mapsto \big((p_n A_n + q_n J)^{-1} + r_n J\big)^{-1}, \tag{1.7}$$

which permits to change the non-symmetric conductivity $\sigma_n(h)$ into a symmetric one. Then, extending the duality principle (1.4) established in [16], we prove that transformation (1.7) is also stable under high-contrast conductivity homogenization.

In the periodic case, i.e. when $\sigma_n(h)(\cdot) = \Sigma_n(\cdot/\varepsilon_n)$ with Σ_n Y-periodic and $\varepsilon_n \to 0$, we use an alternative approach based on an extension of Theorem 4.1 of [13] to $\varepsilon_n Y$-periodic but non-symmetric conductivities (see Theorem 3.1). So, it turns out that the homogenized conductivity $\sigma_*(h)$ is the limit as $n \to \infty$ of the constant H-limit $(\sigma_n)_*$ associated with the periodic homogenization (see, e.g., [2]) of the oscillating sequence $\Sigma_n(\cdot/\varepsilon)$ as $\varepsilon \to 0$ and for a fixed n. Finally, the Dykhne transformation performed by Milton [34] (see also [35], Chapter 4) applied to the local periodic conductivity Σ_n and its effective conductivity $(\sigma_n)_*$, allows us to recover the perturbed homogenized formula (1.6). An example of a periodic cross-like thin structure provides an explicit computation of $\sigma_*(h)$ (see Proposition 3.2).

To make a comparison with dimension three we restrict ourselves to the $\varepsilon_n Y$-periodic fiber-reinforced structure introduced by Fenchenko, Khruslov [24] to derive a nonlocal effect in homogenization. However, in the present context the fiber radius r_n is chosen to be super-critical, i.e. $r_n \to 0$ and $\varepsilon_n^2 |\ln r_n| \to 0$, in order to avoid such an effect. Similarly to (1.5) and following the modelization (1.3), the perturbed conductivity is defined for $h \in \mathbb{R}^3$, by

$$\sigma_n(h) := (1-\chi_n)\big(\alpha_1 I_3 + \beta_1 \mathscr{E}(h)\big) + \chi_n\big(\alpha_{2,n} I_3 + \beta_{2,n} \mathscr{E}(h)\big), \tag{1.8}$$

where χ_n is the characteristic function of the fibers which are parallel to the direction e_3. The form of (1.8) ensures the rotational invariance of $\sigma_n(h)$ for those orthogonal transformations which leave h invariant. Under the same assumptions on the conductivity coefficients as in the two-dimensional case, with $\theta_n = \pi\, r_n^2$, but using a quite different approach, the homogenized conductivity is given by (see Theorem 4.1)

$$\sigma_*(h) = \alpha_1 I_3 + \left(\frac{\alpha_2^3 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2^2 + \beta_2^2 h_3^2}\right) e_3 \otimes e_3 + \beta_1 \mathscr{E}(h). \tag{1.9}$$

The difference between formulas (1.6) and (1.9) provides a new example of gap between dimension two and dimension three in the high-contrast homogenization framework. As former examples of dimensional gap, we refer to the works [17, 20] about the 2d positivity property, versus the 3d non-positivity, of the effective Hall coefficient, and to the works [13, 24] concerning the 2d lack, versus the 3d appearance, of nonlocal effects in the homogenization process.

The paper is organized as follows. Section 2 and 3 deal with dimension two. In Section 2 we study the two-dimensional general (non-periodic) case thanks to an appropriate div-curl lemma. In Section 3 an alternative approach is performed in the periodic framework. Finally, Section 4 is devoted to the three-dimensional case with the fiber-reinforced structure.

Notations

- Ω denotes a bounded open subset of $\mathbb{R}^d$;
- I_d denotes the unit matrix in $\mathbb{R}^{d\times d}$, and $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$;
- for any matrix A in $\mathbb{R}^{d\times d}$, A^{T} denotes the transposed of the matrix A, A^s denotes its symmetric part;
- for $h \in \mathbb{R}^3$, $\mathscr{E}(h)$ denotes the antisymmetric matrix in $\mathbb{R}^{3\times 3}$ defined by $\mathscr{E}(h)\, x := h \times x$, for $x \in \mathbb{R}^3$;
- for any $A, B \in \mathbb{R}^{d\times d}$, $A \le B$ means that for any $\xi \in \mathbb{R}^d$, $A\xi\cdot\xi \le B\xi\cdot\xi$; we will use the fact that for any invertible matrix $A \in \mathbb{R}^{d\times d}$, $A \ge \alpha I_d \Rightarrow A^{-1} \le \alpha^{-1} I_d$;
- $|\cdot|$ denotes both the euclidean norm in $\mathbb{R}^d$ and the subordinate norm in $\mathbb{R}^{d\times d}$;
- for any locally compact subset X of $\mathbb{R}^d$, $\mathcal{M}(X)$ denotes the space of the Radon measures defined on X;

- for any $\alpha, \beta > 0$, $\mathcal{M}(\alpha, \beta; \Omega)$ is the set of the invertible matrix-valued functions $A : \Omega \to \mathbb{R}^{d\times d}$ such that

$$\forall\, \xi \in \mathbb{R}^d, \quad A(x)\xi\cdot\xi \geq \alpha|\xi|^2 \quad \text{and} \quad A^{-1}(x)\xi\cdot\xi \geq \beta^{-1}|\xi|^2 \quad \text{a.e. in } \Omega; \tag{1.10}$$

- C denotes a constant which may vary from a line to another one.

In the sequel, we will use the following extension of H-convergence and introduced in [16]:

Definition 1.1 *Let α_n and β_n be two sequences of positive numbers such that $\alpha_n \leq \beta_n$, and let A_n be a sequence of matrix-valued functions in $\mathcal{M}(\alpha_n, \beta_n; \Omega)$ (see (1.10)).*

The sequence A_n is said to $H(\mathcal{M}(\Omega)^2)$-converge to the matrix-valued function A_* if for any distribution f in $H^{-1}(\Omega)$, the solution u_n of the problem

$$\begin{cases} \operatorname{div}(A_n\nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies the convergences

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_0^1(\Omega) \\ A_n\nabla u_n \rightharpoonup A_*\nabla u & \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2, \end{cases}$$

where u is the solution of the problem

$$\begin{cases} \operatorname{div}(A_*\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We now give a notation for $H(\mathcal{M}(\Omega)^2)$-limits of high-contrast two-phase composites. We consider the characteristic function χ_n of the highly conducting phase, and denote $\omega_n := \{\chi_n = 1\}$.

Notation 1.1 *A sequence of isotropic two-phase conductivities in the absence of a magnetic field is denoted by*

$$\sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) := (1-\chi_n)\alpha_{1,n}I_2 + \chi_n\alpha_{2,n}I_2, \tag{1.11}$$

with

$$\lim_{n\to\infty} \alpha_{1,n} = \alpha_1 > 0 \quad \text{and} \quad \lim_{n\to\infty} |\omega_n|\,\alpha_{2,n} = \alpha_2 > 0, \tag{1.12}$$

and its $H(\mathcal{M}(\Omega)^2)$-limit is denoted by $\sigma_*^0(\alpha_1, \alpha_2)$.

2 A two-dimensional non-periodic medium

2.1 A div-curl approach

We extend the classical div-curl lemma.

Lemma 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^2$. Let $\alpha > 0$, let $\bar{a} \in L^\infty(\Omega)$ and let A_n be a sequence of matrix-valued functions in $L^\infty(\Omega)^{2\times 2}$ (not necessarily symmetric) satisfying*

$$A_n \geq \alpha I_2 \text{ a.e. in } \Omega \quad \text{and} \quad \frac{\det A_n}{\det A_n^s}|A_n^s| \rightharpoonup \bar{a} \in L^\infty(\Omega) \text{ weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{2.1}$$

Let ξ_n be a sequence in $L^2(\Omega)^2$ and v_n a sequence in $H^1(\Omega)$ satisfying the following assumptions:

(i) ξ_n and v_n satisfy the estimate

$$\int_\Omega A_n^{-1}\xi_n\cdot\xi_n\,dx + ||v_n||_{H^1(\Omega)} \leq C; \tag{2.2}$$

(ii) ξ_n satisfies the classical condition

$$\operatorname{div} \xi_n \textit{ is compact in } H^{-1}(\Omega). \tag{2.3}$$

Then, there exist ξ in $L^2(\Omega)^2$ and v in $H^1(\Omega)$ such that the following convergences hold true up to a subsequence

$$\xi_n \rightharpoonup \xi \quad \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega)^2 \quad \textit{ and } \quad \nabla v_n \rightharpoonup \nabla v \quad \textit{weakly in } L^2(\Omega)^2. \tag{2.4}$$

Moreover, we have the following convergence in the distribution sense

$$\xi_n \cdot \nabla v_n \rightharpoonup \xi \cdot \nabla v \quad \textit{weakly in } \mathscr{D}'(\Omega).$$

Proof of Lemma 2.1. The proof consists in considering the "good-divergence" sequence ξ_n as a sum of a compact sequence of gradients ∇u_n and a sequence of divergence-free functions $J\nabla z_n$. We then use Lemma 3.1 of [16] to obtain the strong convergence of z_n in $L^2_{loc}(\Omega)$. Finally, replacing ξ_n by $\nabla u_n + J\nabla z_n$, we conclude owing to integration by parts.

First step: Proof of convergences (2.4).

An easy computation gives

$$\left((A_n^{-1})^s \right)^{-1} = \frac{\det A_n}{\det A_n^s} A_n^s. \tag{2.5}$$

The sequence ξ_n is bounded in $L^1(\Omega)^2$ since the Cauchy-Schwarz inequality combined with the weak-$*$ convergence of (2.1), (2.2) and (2.5) yields

$$\left(\int_\Omega |\xi_n| \, \mathrm{d}x \right)^2 \leq \int_\Omega \left| \left((A_n^{-1})^s \right)^{-1} \right| \, \mathrm{d}x \int_\Omega (A_n^{-1})^s \xi_n \cdot \xi_n \, \mathrm{d}x = \int_\Omega \frac{\det A_n}{\det A_n^s} |A_n^s| \, \mathrm{d}x \int_\Omega A_n^{-1} \xi_n \cdot \xi_n \, \mathrm{d}x \leq C.$$

Therefore, ξ_n converges up to a subsequence to some $\xi \in \mathcal{M}(\Omega)^2$ in the weak-$*$ sense of the measures. Let us prove that the vector-valued measure ξ is actually in $L^2(\Omega)^2$. Again by the Cauchy-Schwarz inequality combined with (2.1), (2.2) and (2.5) we have, for any $\Phi \in \mathscr{C}_0(\Omega)^2$,

$$\begin{aligned} &\left| \int_\Omega \xi(\mathrm{d}x) \cdot \Phi \right| = \lim_{n\to\infty} \left| \int_\Omega \xi_n \cdot \Phi \, \mathrm{d}x \right| \\ &\leq \limsup_{n\to\infty} \left(\int_\Omega \frac{\det A_n}{\det A_n^s} |A_n^s| \, |\Phi|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega A_n^{-1} \xi_n \cdot \xi_n \, \mathrm{d}x \right)^{\frac{1}{2}} \leq C \left(\int_\Omega \bar{a} |\Phi|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that ξ is absolutely continuous with respect to the Lebesgue measure. Since $\bar{a} \in L^\infty(\Omega)$, we also get that

$$\left| \int_\Omega \xi \cdot \Phi \, \mathrm{d}x \right| \leq ||\Phi||_{L^2(\Omega)^2}$$

hence $\xi \in L^2(\Omega)^2$. Therefore, the first convergence of (2.4) holds true with its limit in $L^2(\Omega)^2$. The second one is immediate.

Second step: Introduction of a stream function.

By (2.3), the sequence u_n in $H_0^1(\Omega)$ defined by $u_n := \Delta^{-1} (\operatorname{div} \xi_n)$ strongly converges in $H_0^1(\Omega)$:

$$u_n \longrightarrow u \quad \text{in } H_0^1(\Omega). \tag{2.6}$$

Let ω be a regular simply connected open set such that $\omega \subset\subset \Omega$. Since by definition $\xi_n - \nabla u_n$ is a divergence-free function in $L^2(\Omega)^2$, there exists (see, e.g., [25]) a unique stream function $z_n \in H^1(\omega)$ with zero ω-average such that

$$\xi_n = \nabla u_n + J\nabla z_n \quad \text{a.e. in } \omega. \tag{2.7}$$

Third step: Convergence of the stream function z_n.

Since ∇u_n is bounded in $L^2(\Omega)^2$ by the second step, ξ_n is bounded in $L^1(\Omega)^2$ by the first step and z_n has a zero ω-average, the Sobolev embedding of $W^{1,1}(\omega)$ into $L^2(\omega)$ combined with the Poincaré-Wirtinger inequality in ω implies that z_n is bounded in $L^2(\omega)$ and thus converges, up to a subsequence still denoted by n, to a function z in $L^2(\omega)$.
Moreover, let us define

$$S_n := \left(J^{-1}(A_n^{-1})^s J\right)^{-1}.$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\int_\omega S_n^{-1}\nabla z_n \cdot \nabla z_n \, \mathrm{d}x = \int_\omega J^{-1}(A_n^{-1})^s J\nabla z_n \cdot \nabla z_n \, \mathrm{d}x \\
&= \int_\omega (A_n^{-1})^s J\nabla z_n \cdot J\nabla z_n \, \mathrm{d}x \\
&= \int_\omega (A_n^{-1})^s [\xi_n - \nabla u_n] \cdot [\xi_n - \nabla u_n] \, \mathrm{d}x \\
&\leq 2\int_\omega (A_n^{-1})^s \xi_n \cdot \xi_n \, \mathrm{d}x + 2\int_\omega (A_n^{-1})^s \nabla u_n \cdot \nabla u_n \, \mathrm{d}x \\
&= 2\int_\omega A_n^{-1}\xi_n \cdot \xi_n \, \mathrm{d}x + 2\int_\omega A_n^{-1}\nabla u_n \cdot \nabla u_n \, \mathrm{d}x.
\end{aligned}$$

The first term is bounded by (2.2) and the last term by the inequality $A_n^{-1} \leq \alpha^{-1} I_2$ and the convergence (2.6). Therefore, the sequences $v_n := z_n$ and, by (2.14), S_n satisfy all the assumptions of Lemma 3.1 of [16] since, by (2.5),

$$S_n = \frac{\det A_n}{\det A_n^s} J^{-1} A_n^s J.$$

Then, we obtain the convergence

$$z_n \longrightarrow z \quad \text{strongly in } L^2_{\mathrm{loc}}(\omega). \tag{2.8}$$

Moreover, the convergence (2.6) gives

$$\xi = \nabla u + J\nabla z \quad \text{in } \mathscr{D}'(\omega). \tag{2.9}$$

Fourth step: Integration by parts and conclusion.

We have, as $J\nabla v_n$ is a divergence-free function,

$$\xi_n \cdot \nabla v_n = (\nabla u_n + J\nabla z_n) \cdot \nabla v_n = \nabla u_n \cdot \nabla v_n - \operatorname{div}(z_n J\nabla v_n). \tag{2.10}$$

The strong convergence of ∇u_n in (2.6), the second weak convergence of (2.4) justified in the first step and (2.8) give

$$\nabla u_n \cdot \nabla v_n - \operatorname{div}(z_n J\nabla v_n) \rightharpoonup \nabla u \cdot \nabla v - \operatorname{div}(zJ\nabla v) \quad \text{in } \mathscr{D}'(\omega). \tag{2.11}$$

We conclude, by combining this convergence with (2.10), (2.9) and integrating by parts, to the convergence

$$\xi_n \cdot \nabla v_n \rightharpoonup \nabla u \cdot \nabla v - \operatorname{div}(zJ\nabla v) = (\nabla u + J\nabla z) \cdot \nabla v = \xi \cdot \nabla v \;\; \text{weakly in } \mathscr{D}'(\omega).$$

for an arbitrary open subset ω of Ω. □

For the reader's convenience, we first recall in Theorem 2.1 below the main result of [16] concerning the Keller duality for high contrast conductivities. Then, Proposition 2.1 is an extension of this result to a more general transformation.

Theorem 2.1 ([16]) *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let $\alpha > 0$, let β_n, $n \in \mathbb{N}$ be a sequence of real numbers such that $\beta_n \geq \alpha$, and let A_n be a sequence of matrix-valued functions (not necessarily symmetric) in $\mathcal{M}(\alpha, \beta_n; \Omega)$. Assume that there exists a function $\bar{a} \in L^\infty(\Omega)$ such that*

$$\frac{\det A_n}{\det A_n^s}|A_n^s| \rightharpoonup \bar{a} \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{2.12}$$

Then, there exist a subsequence of n, still denoted by n, and a matrix-valued function A_* in $\mathcal{M}(\alpha, \beta; \Omega)$, with $\beta = 2||\bar{a}||_{L^\infty(\Omega)}$, such that

$$A_n \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} A_* \quad \text{and} \quad \frac{A_n^T}{\det A_n} \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} \frac{A_*^T}{\det A_*}. \tag{2.13}$$

Proposition 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let p_n, q_n and r_n, $n \in \mathbb{N}$ be sequences of real numbers converging respectively to $p > 0$, q and 0. Let $\alpha > 0$, let β_n, $n \in \mathbb{N}$ be a sequence of real numbers such that $\beta_n \geq \alpha$, and let A_n be a sequence of matrix-valued functions in $\mathcal{M}(\alpha, \beta_n; \Omega)$ (not necessarily symmetric) satisfying*

$$r_n A_n \text{ is bounded in } L^\infty(\Omega)^{2\times 2} \quad \text{and} \quad \frac{\det A_n}{\det A_n^s}|A_n^s| \rightharpoonup \bar{a} \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \tag{2.14}$$

and that

$$B_n = \big((p_n A_n + q_n J)^{-1} + r_n J\big)^{-1} \quad \text{is a sequence of symmetric matrices.} \tag{2.15}$$

Then, there exist a subsequence of n, still denoted by n, and a matrix-valued function A_* in $\mathcal{M}(\alpha, \beta; \Omega)$, with $\beta = 2||\bar{a}||_{L^\infty(\Omega)}$, such that

$$A_n \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} A_* \quad \text{and} \quad \big((p_n A_n + q_n J)^{-1} + r_n J\big)^{-1} \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} pA_* + qJ. \tag{2.16}$$

Remark 2.1 *Proposition 2.1 completes Theorem 2.1 performed with the transformation*

$$A \longmapsto \frac{A^T}{\det A} = J^{-1}A^{-1}J, \tag{2.17}$$

to other Dykhne transformations of type (see [35], Section 4.1):

$$A \longmapsto \big((pA + qJ)^{-1} + rJ\big)^{-1} = (pA + qJ)\big((1 - rq)I_2 + rpJA\big)^{-1} \tag{2.18}$$

Remark 2.2 *The convergence of r_n to $r = 0$ is not necessary but sufficient for our purpose. If $r \neq 0$, the different convergences are conserved but lead us to the expression*

$$pA_* + qJ = B_*\big((1 - qr)I_2 + p\, rJA_*\big). \tag{2.19}$$

Proof of Proposition 2.1. The proof is divided into two steps. In the first step, we use Lemma 2.1 to show the $H(\mathcal{M}(\Omega)^2)$-convergence of $\widetilde{A}_n := p_n A_n + q_n J$ to $pA_* + qJ$. In the second step, we build a matrix Q_n which will be used as a corrector for B_n and then use again Lemma 2.1.

First step: $\widetilde{A}_* = pA_* + qJ$.

First of all, thanks to Theorem 2.2 [16], we already know that, up to a subsequence still denoted by n, A_n $H(\mathcal{M}(\Omega)^2)$-converges to A_*. We consider a corrector P_n associated with A_n in the sense of Murat-Tartar (see, e.g., [38]), such that, for $\lambda \in \mathbb{R}^2$, $P_n\lambda = \nabla w_n^\lambda$ is defined by

$$\begin{cases} \operatorname{div}(A_n \nabla w_n^\lambda) = \operatorname{div}(A_* \nabla(\lambda \cdot x)) & \text{in } \Omega \\ w_n^\lambda = \lambda \cdot x & \text{on } \partial\Omega \end{cases} \tag{2.20}$$

Again with Theorem 2.2 of [16] and Definition 1.1, we know that $P_n\lambda$ converges weakly in $L^2(\Omega)^2$ to λ and $A_nP_n\lambda$ converges weakly-$*$ in $\mathcal{M}(\Omega)$ to $A_*\lambda$.

Since, for any $\lambda, \mu \in \mathbb{R}^2$,

$$\alpha||\nabla w_n^\mu||^2_{L^2(\Omega)^2} \leq \int_\Omega A_n\nabla w_n^\mu \cdot \nabla w_n^\mu \, \mathrm{d}x = \int_\Omega A_*\mu \cdot \nabla w_n^\mu \, \mathrm{d}x \leq 2||\bar{a}||_{L^\infty(\Omega)}|\mu| \; |\Omega|^{1/2}||\nabla w_n^\mu||_{L^2(\Omega)^2}$$

and

$$\int_\Omega A_n^{-1}A_n\nabla w_n^\lambda \cdot A_n\nabla w_n^\lambda \, \mathrm{d}x = \int_\Omega A_n\nabla w_n^\lambda \cdot \nabla w_n^\lambda \, \mathrm{d}x,$$

the sequences $\xi_n := A_n\nabla w_n^\lambda$ and $v_n := w_n^\mu$ satisfy (2.2) and (2.3). This combined with (2.14) implies that we can apply Lemma 2.1 to obtain

$$\forall\, \lambda, \mu \in \mathbb{R}, \quad A_nP_n\lambda \cdot P_n\mu \rightharpoonup A_*\lambda \cdot \mu \text{ in } \mathscr{D}'(\Omega). \tag{2.21}$$

We denote $\widetilde{A}_n := p_nA_n + q_nJ$ and consider δ_n such that $\delta_nJ := A_n - A_n^s$. Then, the matrix $\widetilde{A}_n$ satisfies

$$\widetilde{A}_n\xi \cdot \xi = p_nA_n\xi \cdot \xi \geq p_n\alpha|\xi|^2. \tag{2.22}$$

Moreover,

$$\det \widetilde{A}_n = p_n^2 \det A_n^s + (p_n\delta_n + q_n)^2 \leq p_n^2(\det A_n^s + 2\delta_n^2) + 2q_n^2 \leq 2p_n^2 \det A_n + 2q_n^2 \leq C \det A_n,$$

the last inequality being a consequence of $A_n \geq \alpha I_2$. This inequality gives, by (2.14),

$$\frac{\det \widetilde{A}_n}{\det \widetilde{A}_n^s}|\widetilde{A}_n^s| = \frac{\det \widetilde{A}_n}{p_n^2 \det A_n^s}p_n|A_n^s| \leq C\frac{\det A_n}{\det A_n^s}|A_n^s| \leq C. \tag{2.23}$$

Then by (2.22), (2.23) and [16] again, up to a subsequence still denoted by n, $\widetilde{A}_n$ $H(\mathcal{M}(\Omega)^2)$-converges to $\widetilde{A}_*$ and we have, by the classical div-curl lemma of [38] for $JP_n\lambda \cdot P_n\mu$ and (2.21),

$$\forall\, \lambda, \mu \in \mathbb{R}, \quad (p_nA_n + q_nJ)P_n\lambda \cdot P_n\mu = p_nA_nP_n\lambda \cdot P_n\mu + q_nJP_n\lambda \cdot P_n\mu \overset{\mathscr{D}'(\Omega)}{\rightharpoonup} pA^*\lambda \cdot \mu + qJ\lambda \cdot \mu,$$

that can be rewritten

$$\widetilde{A}_* = pA_* + qJ.$$

Second step: $B_* = \widetilde{A}_*$.

Let $\theta \in \mathscr{C}_c^1(\Omega)$ and $\widetilde{P}_n$ a corrector associated with $\widetilde{A}_n$, such that, for $\lambda \in \mathbb{R}^2$, $\widetilde{P}_n\lambda = \nabla\widetilde{w}_n^\lambda$ is defined by

$$\begin{cases} \operatorname{div}(\widetilde{A}_n\nabla\widetilde{w}_n^\lambda) = \operatorname{div}\big(\widetilde{A}_*\nabla(\theta\lambda \cdot x)\big) & \text{in } \Omega \\ \widetilde{w}_n^\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.24}$$

By Definition 1.1, we have

$$\begin{cases} \widetilde{w}_n^\lambda & \rightharpoonup \theta\lambda \cdot x & \text{weakly in } H_0^1(\Omega), \\ \widetilde{A}_n\nabla\widetilde{w}_n^\lambda & \rightharpoonup \widetilde{A}_*\nabla(\theta\lambda \cdot x) & \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2. \end{cases} \tag{2.25}$$

Let us now consider $B_n = \big(\widetilde{A}_n^{-1} + r_nJ\big)^{-1}$. B_n is symmetric and so is its inverse :

$$B_n^{-1} = \widetilde{A}_n^{-1} + r_nJ = (\widetilde{A}_n^{-1} + r_nJ)^s = (\widetilde{A}_n^{-1})^s.$$

We then have, by a little computation (like in (2.5)) and (2.23),

$$\frac{\det B_n}{\det B_n^s}|B_n^s| = |B_n| = \left|\left((\widetilde{A}_n^{-1})^s\right)^{-1}\right| = \frac{\det \widetilde{A}_n}{\det \widetilde{A}_n^s}|\widetilde{A}_n^s| \leq C. \tag{2.26}$$

For any $\xi \in \mathbb{R}^2$, the sequence $\nu_n := (I + r_n J\widetilde{A}_n)^{-1}\xi$ satisfies, by (2.14),

$$|\xi| \leq \left(1 + ||r_n\widetilde{A}_n||_{L^\infty(\Omega)^{2\times 2}}\right)|\nu_n| \leq \left(1 + p_n||r_n A_n||_{L^\infty(\Omega)^{2\times 2}} + q_n r_n\right)|\nu_n| \leq (1 + C)|\nu_n|,$$

hence

$$B_n\xi\cdot\xi = \widetilde{A}_n\nu_n\cdot(I+r_nJ\widetilde{A}_n)\nu_n = \widetilde{A}_n\nu_n\cdot\nu_n = p_nA_n\nu_n\cdot\nu_n \geq p_n\alpha|\nu_n|^2 \geq \alpha\frac{p_n}{(1+C)^2}|\xi|^2 \geq C|\xi|^2 \quad (2.27)$$

with $C > 0$. Therefore, with (2.27) and (2.26), again by Theorem 2.2 of [16], up to a subsequence still denoted by n, B_n $H(\mathcal{M}(\Omega)^2)$-converges to B_*.

Let $\psi \in \mathscr{C}^1_c(\Omega)$ and R_n be a corrector associated to B_n, such that, for $\mu \in \mathbb{R}^2$, $R_n\mu = \nabla v_n^\mu$ is defined by

$$\begin{cases} \operatorname{div}(B_n\nabla v_n^\mu) = \operatorname{div}\big(B_*\nabla(\psi\mu\cdot x)\big) & \text{in } \Omega \\ v_n^\mu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.28}$$

By Definition 1.1, we have the convergences

$$\begin{cases} v_n^\mu & \rightharpoonup \psi\mu\cdot x & \text{weakly in } H^1_0(\Omega), \\ B_n\nabla v_n^\mu & \rightharpoonup B_*\nabla(\psi\mu\cdot x) & \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2. \end{cases} \tag{2.29}$$

Let us define the matrix $Q_n := (I + r_nJ\widetilde{A}_n)\widetilde{P}_n$. We have

$$B_nQ_n = (\widetilde{A}_n^{-1} + r_nJ)^{-1}(I + r_nJ\widetilde{A}_n)\widetilde{P}_n = (\widetilde{A}_n^{-1} + r_nJ)^{-1}(\widetilde{A}_n^{-1} + r_nJ)\widetilde{A}_n\widetilde{P}_n = \widetilde{A}_n\widetilde{P}_n. \tag{2.30}$$

We are going to pass to the limit in $\mathscr{D}'(\Omega)$ the equality given by (2.30) and the symmetry of B_n:

$$\widetilde{A}_n\widetilde{P}_n\lambda\cdot R_n\mu = B_nQ_n\lambda\cdot R_n\mu = Q_n\lambda\cdot B_nR_n\mu. \tag{2.31}$$

On the one hand, $\widetilde{A}_n$ satisfies (2.1) by (2.22) and (2.23). The sequences $\xi_n := \widetilde{A}_n\widetilde{P}_n\lambda$ and $v_n := v_n^\mu$ satisfy the hypothesis (2.3) by (2.24) and (2.2) because

$$\int_\Omega (\widetilde{A}_n)^{-1}\xi_n\cdot\xi_n \, dx + ||v_n||_{H^1_0(\Omega)} = \int_\Omega \widetilde{A}_n\widetilde{P}_n\lambda\cdot\widetilde{P}_n\lambda \, dx + ||v_n^\mu||_{H^1_0(\Omega)} \, dx \leq C$$

by (2.24) and the convergences (2.29) and (2.25). The application of Lemma 2.1, (2.25) and (2.29) give the convergence

$$\widetilde{A}_n\widetilde{P}_n\lambda\cdot R_n\mu \rightharpoonup A^*\nabla(\theta\lambda\cdot x)\cdot\nabla(\psi\mu\cdot x) \quad \text{in } \mathscr{D}'(\Omega). \tag{2.32}$$

On the other hand, we have the equality

$$Q_n\lambda\cdot B_nR_n\mu = B_nR_n\mu\cdot\widetilde{P}_n\lambda + B_nR_n\mu\cdot r_nJ\widetilde{A}_n\widetilde{P}_n. \tag{2.33}$$

The matrix B_n satisfies (2.1) by (2.27) and (2.26). The sequences $\xi_n := B_nR_n\mu$ and $v_n := \widetilde{w}_n^\lambda$ satisfy the hypothesis (2.3) by (2.28) and (2.2) of Lemma 2.1 because

$$\int_\Omega (B_n)^{-1}\xi_n\cdot\xi_n \, dx + ||v_n||_{H^1_0(\Omega)} = \int_\Omega B_nR_n\mu\cdot R_n\mu \, dx + ||\widetilde{w}_n^\lambda||_{H^1_0(\Omega)} \, dx \leq C$$

by (2.28) and the convergences (2.25) and (2.29). The application of Lemma 2.1, (2.25) and (2.29) give the convergence

$$B_nR_n\mu\cdot\widetilde{P}_n\lambda \rightharpoonup B_*\nabla(\psi\mu\cdot x)\cdot\nabla(\theta\lambda\cdot x) \quad \text{in } \mathscr{D}'(\Omega). \tag{2.34}$$

The convergence of the right part of (2.33) is more delicate. The demonstration is the same as for Lemma 2.1. Let ω be a simply connected open subset of Ω such as $\omega \subset\subset \Omega$. The function $\widetilde{A}_n\widetilde{P}_n\lambda - \widetilde{A}_*\nabla(\theta\lambda\cdot x)$ is divergence-free and we can introduce a function z_n^λ such as

$$\widetilde{A}_n\widetilde{P}_n\lambda = \widetilde{A}_*\nabla(\theta\lambda\cdot x) + J\nabla z_n^\lambda, \tag{2.35}$$

$$z_n^\lambda \longrightarrow 0 \quad \text{strongly in } L^2_{\text{loc}}(\omega). \tag{2.36}$$

The equality

$$\begin{aligned} B_nR_n\mu\cdot r_nJ\widetilde{A}_n\widetilde{P}_n\lambda &= r_nB_nR_n\mu\cdot J\widetilde{A}_*\nabla(\theta\lambda\cdot x) - r_nB_nR_n\mu\cdot\nabla z_n^\lambda \\ &= r_nB_nR_n\mu\cdot J\widetilde{A}_*\nabla(\theta\lambda\cdot x) - r_n\text{div}(z_n^\lambda B_nR_n\mu) + r_nz_n^\lambda\,\text{div}\big(B_*\nabla(\theta\lambda\cdot x)\big) \end{aligned}$$

leads us, by (2.29), (2.36) and the convergence to 0 of r_n, like in the demonstration of Lemma 2.1, to

$$B_nR_n\mu\cdot r_nJ\widetilde{A}_n\widetilde{P}_n \rightharpoonup 0 \quad \text{in } \mathscr{D}'(\omega). \tag{2.37}$$

Finally, by combining (2.31), (2.32), (2.34) and (2.37), we obtain, for any simply connected open subset ω of Ω such as $\omega \subset\subset \Omega$,

$$\widetilde{A}_*\nabla(\theta\lambda\cdot x)\cdot\nabla(\psi\mu\cdot x) = B_*\nabla(\psi\mu\cdot x)\cdot\nabla(\theta\lambda\cdot x) \quad \text{in } \mathscr{D}'(\omega).$$

We conclude, by taking $\theta = 1$ and $\psi = 1$ on ω and taking into account that B_* is symmetric and ω, λ, μ are arbitrary, that:

$$B_* = \widetilde{A}_* = pA_* + qJ.$$

□

2.2 An application to isotropic two-phase media

In this section, we study the homogenization of a two-phase isotropic medium with high contrast and non-necessarily symmetric conductivities. The study of the symmetric case in Proposition 2.2 permits to obtain Theorem 2.2 by applying the transformation of Proposition 2.1. We use Notation 1.1.

Proposition 2.2 *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let ω_n, n in $\mathbb{N}$, be a sequence of open subsets of Ω with characteristic function χ_n, satisfying $\theta_n := |\omega_n| < 1$, θ_n converges to 0, and*

$$\frac{\chi_n}{\theta_n} \rightharpoonup a \in L^\infty(\Omega) \quad \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega). \tag{2.38}$$

We assume that there exists $\alpha_1, \alpha_2 > 0$ and two positive sequences $\alpha_{1,n}$, $\alpha_{2,n} \geq a_0 > 0$ verifying

$$\lim_{n\to\infty}\alpha_{1,n} = \alpha_1 \quad \textit{and} \quad \lim_{n\to\infty}\theta_n\alpha_{2,n} = \alpha_2, \tag{2.39}$$

and that the conductivity takes the form

$$\sigma_n^0(\alpha_{1,n},\alpha_{2,n}) = (1-\chi_n)\alpha_{1,n}I_2 + \chi_n\alpha_{2,n}I_2.$$

Then, there exists a subsequence of n, still denoted by n, and a locally Lipschitz function

$$\sigma_*^0 : (0,\infty)^2 \longrightarrow \mathcal{M}(a_0, 2||a||_\infty;\Omega)$$

such that

$$\forall\,(\alpha_1,\alpha_2)\in(0,\infty)^2, \quad \sigma_n^0(\alpha_{1,n},\alpha_{2,n}) \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} \sigma_*^0(\alpha_1,\alpha_2). \tag{2.40}$$

Proof of Proposition 2.2. The proof is divided into two parts. We first prove the theorem for $\alpha_{1,n} = \alpha_1$, $\alpha_{2,n} = \theta_n^{-1}\alpha_2$, and then treat the general case.

First step: The case $\alpha_{1,n} = \alpha_1$, $\alpha_{2,n} = \theta_n^{-1}\alpha_2$.
In this step we denote $\sigma_n^0(\alpha) := \sigma_n^0(\alpha_1, \theta_n^{-1}\alpha_2)$, for $\alpha = (\alpha_1, \alpha_2) \in (0,\infty)^2$. Theorem 2.2 of [16] implies that for any $\alpha \in (0,\infty)^2$, there exists a subsequence of n such that $\sigma_n^0(\alpha)$ $H(\mathcal{M}(\Omega)^2)$-converges in the sense of Definition 1.1 to some matrix-valued function in $\mathcal{M}(a_0, 2\|a\|_\infty; \Omega)$.

By a diagonal extraction, there exists a subsequence of n, still denoted by n, such that

$$\forall\, \alpha \in \mathbb{Q}^2 \cap (0,\infty)^2, \quad \sigma_n^0(\alpha) \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} \sigma_*^0(\alpha). \tag{2.41}$$

We are going to show that this convergence is true any pair $\alpha \in (0,\infty)^2$.

We have, by (2.38), for any $\alpha \in \mathbb{Q}^2 \cap (0,\infty)^2$,

$$|\sigma_n^0(\alpha)| = (1-\chi_n)\alpha_1 + \chi_n \frac{\alpha_2}{\theta_n} \rightharpoonup \alpha_1 + \alpha_2\, a \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega) \tag{2.42}$$

and, since $\theta_n \in (0,1)$,

$$\forall\, \xi \in \mathbb{R}^2, \quad \sigma_n^0(\alpha)\xi\cdot\xi = \alpha_1(1-\chi_n)|\xi|^2 + \chi_n\frac{\alpha_2}{\theta_n}|\xi|^2 \ge \min(\alpha_1,\alpha_2)|\xi|^2 \quad \text{a.e. in } \Omega. \tag{2.43}$$

By applying Theorem 2.2 of [16] with (2.42), we have the inequality

$$|\sigma_*^0(\alpha)\lambda| \le 2|\lambda|\left(\alpha_1 + \alpha_2\|a\|_\infty\right). \tag{2.44}$$

For any $\alpha \in \mathbb{Q}^2 \cap (0,\infty)^2$ and $\lambda \in \mathbb{R}^2$, consider the corrector $w_n^{\alpha,\lambda}$ associated with $\sigma_n^0(\alpha)$ defined by

$$\begin{cases} \operatorname{div}\left(\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\right) &= \operatorname{div}\left(\sigma_*^0(\alpha)\lambda\right) & \text{in } \Omega, \\ w_n^{\alpha,\lambda} &= \lambda\cdot x & \text{on } \partial\Omega, \end{cases} \tag{2.45}$$

which depends linearly on λ.

Let $\alpha \in \mathbb{Q}^2 \cap (0,\infty)^2$. Let us show that the energies

$$\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\, dx \tag{2.46}$$

are bounded. We have, by (2.45), (2.44) and the Cauchy-Schwarz inequality

$$\begin{aligned} &\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\, dx \\ &= \int_\Omega \sigma_*^0(\alpha)\lambda\cdot\left(\nabla w_n^{\alpha,\lambda} - \lambda\right) dx + \int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda\, dx \\ &= \int_\Omega \sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}\, dx - \int_\Omega \underbrace{\sigma_*^0(\alpha)\lambda\cdot\lambda}_{\ge 0}\, dx + \int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda\, dx \end{aligned}$$

which leads us to

$$\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\, dx \le \int_\Omega |\sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}|\, dx + \int_\Omega |\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda|\, dx. \tag{2.47}$$

On the one hand, the Cauchy-Schwarz inequality gives

$$\left(\int_\Omega |\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda|\, dx\right)^2 \le |\lambda|^2 \int_\Omega |\sigma_n^0(\alpha)|\, dx \int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\, dx$$

that is

$$\left(\int_\Omega |\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda|\,\mathrm{d}x\right)^2 \le |\lambda|^2|\alpha|\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,\mathrm{d}x. \tag{2.48}$$

On the other hand, by (2.43) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_\Omega |\sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}|\,\mathrm{d}x &\le 2|\lambda|(\alpha_1+\alpha_2||a||_\infty)\sqrt{\int_\Omega |\nabla w_n^{\alpha,\lambda}|^2\,\mathrm{d}x}\\ &\le 2|\lambda|(\alpha_1+\alpha_2||a||_\infty)\sqrt{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\sqrt{\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,\mathrm{d}x}\end{aligned}$$

that is

$$\int_\Omega |\sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}|\,\mathrm{d}x \le C\,|\lambda|^2\,|\alpha|\sqrt{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\sqrt{\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,\mathrm{d}x} \tag{2.49}$$

where C does not depend on n nor α.

By combining (2.47), (2.48) and (2.49), we have

$$\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,\mathrm{d}x \le C\,|\lambda|^2\,\big(\underbrace{|\alpha|+|\alpha|^2({\alpha_1}^{-1}+{\alpha_2}^{-1})}_{=:M(\alpha)}\big) \tag{2.50}$$

where C does not depend on n nor α.

Let $\alpha'\in\mathbb{Q}^2\cap(0,\infty)^2$. The sequences $\xi_n := \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}$ and $v_n := w_n^{\alpha',\lambda}$ satisfy the assumptions (2.2) and (2.3) of Lemma 2.1. By symmetry, we have the convergences

$$\begin{cases}\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} & \rightharpoonup \sigma_*^0(\alpha)\lambda\cdot\lambda \quad \text{weakly in } \mathscr{D}'(\Omega),\\ \sigma_n^0(\alpha')\nabla w_n^{\alpha',\lambda}\cdot\nabla w_n^{\alpha,\lambda} & \rightharpoonup \sigma_*^0(\alpha')\lambda\cdot\lambda \quad \text{weakly in } \mathscr{D}'(\Omega).\end{cases} \tag{2.51}$$

As the matrices are symmetric, we have

$$\left(\sigma_n^0(\alpha)-\sigma_n^0(\alpha')\right)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} = \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} - \sigma_n^0(\alpha')\nabla w_n^{\alpha',\lambda}\cdot\nabla w_n^{\alpha,\lambda},$$

hence

$$\left(\sigma_n^0(\alpha)-\sigma_n^0(\alpha')\right)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} \rightharpoonup \left(\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right)\lambda\cdot\lambda \quad \text{weakly in } \mathscr{D}'(\Omega). \tag{2.52}$$

Let $\lambda\in\mathbb{R}^2$. We have, by the Cauchy-Schwarz inequality, with the Einstein convention

$$\begin{aligned}&\int_\Omega \left|(\sigma_n^0(\alpha)-\sigma_n^0(\alpha'))\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right|\,\mathrm{d}x\\ &= \int_{\Omega\setminus\omega_n}|\alpha_1-\alpha_1'|\,\left|\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right|\,\mathrm{d}x + \fint_{\omega_n}|\alpha_2-\alpha_2'|\,\left|\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right|\,\mathrm{d}x\\ &\le |\alpha_1-\alpha_1'|\sqrt{\int_{\Omega\setminus\omega_n}|\nabla w_n^{\alpha,\lambda}|^2\,\mathrm{d}x}\,\sqrt{\int_{\Omega\setminus\omega_n}|\nabla w_n^{\alpha',\lambda}|^2\,\mathrm{d}x}\\ &\qquad + |\alpha_2-\alpha_2'|\sqrt{\fint_{\omega_n}|\nabla w_n^{\alpha,\lambda}|^2\,\mathrm{d}x}\,\sqrt{\fint_{\omega_n}|\nabla w_n^{\alpha',\lambda}|^2\,\mathrm{d}x}\\ &\le |\alpha_i-\alpha_i'|\sqrt{\frac{1}{\alpha_i}\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,\mathrm{d}x}\,\sqrt{\frac{1}{\alpha_i'}\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha',\lambda}\cdot\nabla w_n^{\alpha',\lambda}\,\mathrm{d}x}.\end{aligned}$$

This combined with (2.50) yields

$$\int_\Omega \left|(\sigma_n^0(\alpha)-\sigma_n^0(\alpha'))\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right| \le C|\lambda|^2\,\frac{|\alpha_i-\alpha_i'|}{\sqrt{|\alpha_i||\alpha_i'|}}\,M(\alpha)\,M(\alpha')$$

The sequence of (2.52) is thus bounded in $L^1(\Omega)^2$ which implies that (2.52) holds weakly-$*$ in $\mathcal{M}(\Omega)$. Hence, we get, for any $\varphi \in \mathscr{C}_c(\Omega)$, that

$$\int_\Omega \left|\left(\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right)\lambda\cdot\lambda\right|\ \varphi\, dx \le C\,|\lambda|^2\,\frac{|\alpha_i-\alpha_i'|}{\sqrt{|\alpha_i||\alpha_i'|}}\ M(\alpha)\ M(\alpha')\ ||\varphi||_\infty. \tag{2.53}$$

Then, the Riesz representation theorem implies that

$$\left|\left|\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right|\right|_{L^1(\Omega)^{2\times 2}} \le C\,\frac{|\alpha_i-\alpha_i'|}{\sqrt{|\alpha_i||\alpha_i'|}}\ M(\alpha)\ M(\alpha').$$

Therefore, by the definition of M in (2.50), for any compact subset $K\subset(0,\infty)^2$,

$$\exists\, C>0,\quad \forall\,\alpha,\alpha'\in\mathbb{Q}^2\cap K,\quad \left|\left|\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right|\right|_{L^1(\Omega)^{2\times 2}}\le C\,|\alpha-\alpha'|. \tag{2.54}$$

This estimate permits to extend the definition (2.41) of σ_*^0 on $(0,\infty)^2$ by

$$\forall\,\alpha\in(0,\infty)^2,\quad \sigma_*^0(\alpha)=\lim_{\substack{\alpha'\to\alpha\\ \alpha'\in\mathbb{Q}^2\cap(0,\infty)^2}}\sigma_*^0(\alpha')\quad\text{strongly in } L^1(\Omega)^{2\times 2}. \tag{2.55}$$

Let $\alpha\in(0,\infty)^2$. Theorem 2.2 of [16] implies that there exists a subsequence of n, denoted by n', and a matrix-valued function $\widetilde{\sigma}_*\in\mathcal{M}(a_0,2||a||_\infty;\Omega)$ such that

$$\sigma_{n'}(\alpha)\ \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup}\ \widetilde{\sigma}_*. \tag{2.56}$$

Repeating the arguments leading to (2.54), for any positive sequence of rational pair $(\alpha^q)_{q\in\mathbb{N}}$ converging to α, we have

$$\exists\, C>0,\quad \left|\left|\widetilde{\sigma}_*-\sigma_*^0(\alpha^q)\right|\right|_{L^1(\Omega)^{2\times 2}}\le C\,|\alpha-\alpha^q|, \tag{2.57}$$

hence, by (2.55), $\widetilde{\sigma}_*=\sigma_*^0(\alpha)$. Therefore by the uniqueness of the limit in (2.56), we obtain for the whole sequence satisfying (2.41)

$$\forall\,\alpha\in(0,\infty)^2,\quad \sigma_n(\alpha)\ \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup}\ \sigma_*^0(\alpha). \tag{2.58}$$

In particular, the function σ_*^0 satisfies (2.54) and (2.55), i.e. σ_*^0 is a locally Lipschitz function on $(0,\infty)^2$.

Second step: The general case.
We denote $\alpha^n=(\alpha_{1,n},\alpha_{2,n})$ and $\sigma_n^0(\alpha^n)=\sigma_n^0(\alpha_{1,n},\alpha_{2,n})$. Theorem 2.2 of [16] implies that there exists a subsequence of n, denoted by n', such that $\sigma_{n'}^0(\alpha^{n'})$ $H(\mathcal{M}(\Omega)^2)$-converges to some $\widetilde{\sigma}_*\in\mathcal{M}(a_0,2||a||_\infty;\Omega)$ in the sense of Definition 1.1.

As in the first step, for any $\alpha^{n'}\in(0,\infty)^2$ and $\lambda\in\mathbb{R}^2$, we can consider the corrector $w_{n'}^{\alpha^{n'},\lambda}$ associated with $\sigma_{n'}^0(\alpha^{n'})$ defined by

$$\begin{cases} \operatorname{div}\left(\sigma_{n'}^0(\alpha^{n'})\nabla w_{n'}^{\alpha^{n'},\lambda}\right) &= \operatorname{div}(\widetilde{\sigma}_*\lambda) \quad \text{in } \Omega,\\ \qquad\qquad w_{n'}^{\alpha^{n'},\lambda} &= \lambda\cdot x \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{2.59}$$

which depends linearly on λ. Proceeding as in the first step, we obtain like in (2.52), with $\alpha=(\alpha_1,\alpha_2)$ the limit of α^n according to (2.39),

$$\left(\sigma_{n'}^0(\alpha)-\sigma_{n'}^0(\alpha^{n'})\right)\nabla w_{n'}^{\alpha^{n'},\lambda}\cdot\nabla w_{n'}^{\alpha,\lambda}\ \rightharpoonup\ \left(\sigma_*^0(\alpha)-\widetilde{\sigma}_*\right)\lambda\cdot\lambda\quad\text{weakly in } \mathscr{D}'(\Omega). \tag{2.60}$$

Moreover, by the energy bound (2.50), which also holds for $\alpha^{n'}$, we have, for any $\varphi \in \mathscr{D}(\Omega)$,

$$\int_\Omega \left(\sigma^0_{n'}(\alpha) - \sigma^0_{n'}(\alpha^{n'})\right) \nabla w^{\alpha^{n'},\lambda}_{n'} \cdot \nabla w^{\alpha,\lambda}_{n'} \, \varphi \, dx \underset{n'\to\infty}{\longrightarrow} 0.$$

This combined with (2.60), yields

$$\int_\Omega \left(\sigma^0_*(\alpha) - \widetilde{\sigma}_*\right) \lambda \cdot \lambda \, \varphi \, dx = 0,$$

which implies that $\sigma^0_*(\alpha) = \widetilde{\sigma}_*$. We conclude by a uniqueness argument. □

We can now obtain a result for (perturbed) non-symmetric conductivities. Then, we will use a Dykhne transformation to recover the symmetric case following the Milton approach [35] (pp. 61–65). This will allow us to apply Proposition 2.2.

Theorem 2.2 *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let ω_n, $n \in \mathbb{N}$, be a sequence of open subsets of Ω and denote by χ_n their characteristic function. We assume that $\theta_n = |\omega_n| < 1$ converges to 0 and*

$$\frac{\chi_n}{\theta_n} \rightharpoonup a \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{2.61}$$

Consider the conductivity defined by

$$\sigma_n(h) = (1-\chi_n)\sigma_1(h) + \frac{\chi_n}{\theta_n}\sigma_2(h) \tag{2.62}$$

where for $j = 1, 2$, $\sigma_j(h) = \alpha_j + h\beta_j J \in \mathbb{R}^{2\times 2}$ with $\alpha_1, \alpha_2 > 0$ and $(\beta_1, \beta_2) \neq (0, 0)$.
Then, there exists a subsequence of n, still denoted by n, and a locally Lipschitz function

$$\sigma^0_* : (0,\infty)^2 \longrightarrow \mathcal{M}\Big(\min(\alpha_1,\alpha_2), 2\big(|\sigma_1| + |\sigma_2| \; ||a||_\infty\big); \Omega\Big)$$

such that

$$\sigma_n(h) \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} \sigma^0_*\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2\big) + h\beta_1 J.$$

Proof of Theorem 2.2. We have

$$\forall\, \xi \in \mathbb{R}^2, \quad \sigma_n(h)\xi \cdot \xi = (1-\chi_n)\alpha_1|\xi|^2 + \frac{\chi_n}{\theta_n}\alpha_2|\xi|^2 \geq \min(\alpha_1,\alpha_2)|\xi|^2 \quad \text{a.e. in } \Omega$$

and, by (2.61),

$$|\sigma_n(h)| = (1-\chi_n)|\sigma_1(h)| + \frac{\chi_n}{\theta_n}|\sigma_2(h)| \rightharpoonup |\sigma_1(h)| + a|\sigma_2(h)| \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

In order to make a Dykhne transformation like in p.62 of [35], we consider two real coefficients a_n and b_n in such a way that

$$B_n := \big(a_n\sigma_n(h) + b_n J\big)\big(a_n I_2 + J\sigma_n(h)\big)^{-1} = \big((p_n\sigma_n(h) + q_n J)^{-1} + r_n J\big)^{-1}$$

is symmetric. An easy computation shows that the previous equality holds when

$$p_n := \frac{a_n^2}{a_n^2 + b_n}, \quad q_n := \frac{a_n b_n}{a_n^2 + b_n} \quad \text{and} \quad r_n := \frac{1}{a_n}.$$

On the one hand, the estimates (3.39) and (3.40) with $\alpha_{2,n} = \theta_n^{-1}\alpha_2$, $\beta_{2,n} = \theta_n^{-1}\beta_2$, yield (note that they are independent of χ_n)

$$p_n \underset{n\to\infty}{\sim} 1, \quad q_n \underset{n\to\infty}{\longrightarrow} -h\beta_1, \quad r_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad ||r_n\sigma_n(h)||_\infty \leq C\big(|\sigma_1(h)| + |\sigma_2(h)|\big). \tag{2.63}$$

On the other hand, as in Section 3.2, with Notation 1.1 and (3.34), we have

$$B_n = \sigma_n^0\big(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\big), \tag{2.64}$$

where

$$\alpha'_{1,n}(h) = \frac{a_n(\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \quad \text{and} \quad \alpha'_{2,n}(h) = \frac{a_n(\alpha_2/\theta_n + ih\beta_2/\theta_n) + ib_n}{a_n + i(\alpha_2/\theta_n + ih\beta_2/\theta_n)}. \tag{2.65}$$

Hence, like in (3.41), we have

$$\lim_{n\to\infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \lim_{n\to\infty} \theta_n \alpha'_{2,n}(h) = \alpha_2 + \alpha_2^{-1}{\beta_2}^2 h^2. \tag{2.66}$$

We can first apply Proposition 2.2 with the conditions (2.64) and (2.66) to have the $H(\mathcal{M}(\Omega)^2)$-convergence of B_n. Then, by virtue of Proposition 2.1, with (2.63) we get that

$$\sigma_n(h) \overset{H(\mathcal{M}(\Omega)^2)}{\longrightarrow} \sigma_*^0\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2\big) + h\beta_1 J.$$

□

3 A two-dimensional periodic medium

In this section we consider a sequence Σ_n of matrix valued functions (not necessarily symmetric) in $L^\infty(\mathbb{R}^2)^{2\times 2}$, which satisfies the following assumptions:

1 . Σ_n is Y-periodic, where $Y := (0,1)^2$, i.e.,

$$\forall\, n \in \mathbb{N}, \forall\, \kappa \in \mathbb{Z}^2, \quad \Sigma_n(. + \kappa) = \Sigma_n(.) \ \text{ a.e. in } \mathbb{R}^2, \tag{3.1}$$

2 . Σ_n is equi-coercive in $\mathbb{R}^2$, i.e.,

$$\exists\, \alpha > 0 \quad \text{such that} \quad \forall\, n \in \mathbb{N}, \forall\, \xi \in \mathbb{R}^2, \quad \Sigma_n\xi \cdot \xi \geq \alpha|\xi|^2 \ \text{ a.e. in } \mathbb{R}^2. \tag{3.2}$$

Let ε_n be a sequence of positive numbers which tends to 0. From the sequences Σ_n and ε_n we define the highly oscillating sequence of matrix-valued functions σ_n by

$$\sigma_n(x) = \Sigma_n\left(\frac{x}{\varepsilon_n}\right), \quad \text{a.e. } x \in \mathbb{R}^2. \tag{3.3}$$

By virtue of (3.1) and (3.2), σ_n is an equi-coercive sequence of ε_n-periodic matrix-valued functions in $L^\infty(\mathbb{R}^2)^{2\times 2}$. For a fixed $n \in \mathbb{N}$, let $(\sigma_n)_*$ be the constant matrix defined by

$$\forall\, \lambda, \mu \in \mathbb{R}^2, \quad (\sigma_n)_*\lambda \cdot \mu = \int_Y \Sigma_n \nabla W_n^\lambda \cdot \nabla W_n^\mu \, dy, \tag{3.4}$$

where, for any $\lambda \in \mathbb{R}^2$, $W_n^\lambda \in H^1_\sharp(Y)$, the set of Y-periodic functions belonging to $H^1_{loc}(\mathbb{R}^2)$, is the solution of the auxiliary problem

$$\int_Y \big(W_n^\lambda - \lambda \cdot y\big)\, dy = 0 \quad \text{and} \quad \operatorname{div}\big(\Sigma_n \nabla W_n^\lambda\big) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^2) \tag{3.5}$$

or equivalently

$$\left\{ \begin{aligned} &\int_Y \Sigma_n \nabla W_n^\lambda \cdot \nabla\varphi \, dy = 0, \quad \forall\, \varphi \in H^1_\sharp(Y) \\ &\int_Y \big(W_n^\lambda(y) - \lambda \cdot y\big)\, dy = 0. \end{aligned} \right. \tag{3.6}$$

Set

$$w_n^\lambda(x) := \varepsilon_n W_n^\lambda\left(\frac{x}{\varepsilon_n}\right), \quad \text{for } x \in \Omega, \tag{3.7}$$

and

$$w_n := (w_n^{e_1}, w_n^{e_2}) = (w_n^1, w_n^2). \tag{3.8}$$

3.1 A uniform convergence result

Theorem 3.1 *Let Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz boundary. Consider a highly oscillating sequence of matrix-valued functions σ_n satisfying (3.1), (3.2), (3.3) and the constant matrix $(\sigma_n)_*$ defined by (3.4). We assume that*

$$(\sigma_n)_* \longrightarrow \sigma_* \text{ in } \mathbb{R}^{2\times 2}. \tag{3.9}$$

Consider, for $f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)$ with $q > 2$, the solution u_n of the problem

$$\mathcal{P}_n \begin{cases} -\operatorname{div}(\sigma_n \nabla u_n) &= f \quad \text{in } \Omega \\ u_n &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Then, u_n converges uniformly to the solution $u \in H_0^1(\Omega)$ of

$$\mathcal{P} \begin{cases} -\operatorname{div}(\sigma_* \nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Moreover we have the corrector result, with the $\varepsilon_n Y$-periodic sequence w_n defined in (3.8):

$$\nabla u_n - \sum_{i=1}^{2} \partial_i u \, \nabla w_n^i \longrightarrow 0 \quad \text{in } L^1(\Omega)^2. \tag{3.12}$$

Remark 3.1 *The first point of Theorem 3.1 is an extension to the non-symmetric case of the results of [13] and [15]. The uniform convergence of u_n is a straightforward consequence of Theorem 2.7 of [15] taking into account that in the present case $\sigma_n \in L^\infty(\Omega)^{2\times 2}$ for a fixed n. The fact that $f \in W^{-1,q}(\Omega)$ with $q > 2$ ensures the uniform convergence.*

Proof of Theorem 3.1.

Derivation of the limit problem $\mathcal{P}$.

We only have to show that u is the solution of $\mathcal{P}$ in (3.11). We consider a corrector $D\widetilde{w}_n : \mathbb{R}^2 \longrightarrow \mathbb{R}^{2\times 2}$ associated with σ_n^{T} defined by

$$\widetilde{w}_n(x) := \varepsilon_n \widetilde{W}_n\left(\frac{x}{\varepsilon_n}\right) = \left(\varepsilon_n \widetilde{W}_n^1\left(\frac{x}{\varepsilon_n}\right), \varepsilon_n \widetilde{W}_n^2\left(\frac{x}{\varepsilon_n}\right)\right)$$

where for $i = 1, 2$, $\widetilde{W}_n^i \in H_\sharp^1(Y)$ is the solution of the auxiliary problem

$$\int_Y \left(\widetilde{W}_n^i - e_i \cdot x\right) \mathrm{d}x = 0 \quad \text{and} \quad \operatorname{div}\left(\Sigma_n^{\mathrm{T}} \nabla \widetilde{W}_n^i\right) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^2). \tag{3.13}$$

Again, thanks to Theorem 2.7 of [15], $\widetilde{w}_n$ converges uniformly to the identity in Ω by the integral condition (3.13). Let $\varphi \in \mathscr{D}(\Omega)$. We have, using the Einstein convention, by integrating by parts and by the Schwarz theorem ($\partial^2_{i,j}\varphi = \partial^2_{j,i}\varphi$)

$$\begin{aligned}
&\int_\Omega \sigma_n \nabla u_n \cdot \nabla\left(\varphi(\widetilde{w}_n)\right)\, \mathrm{d}x \\
&= \int_\Omega \nabla u_n \cdot \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^i (\partial_i \varphi)(\widetilde{w}_n)\, \mathrm{d}x \\
&= \underbrace{\int_\Omega \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^i \cdot \nabla(u_n \partial_i \varphi(\widetilde{w}_n))\, \mathrm{d}x}_{=0} - \int_\Omega \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \ \ \partial^2_{i,j}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x \\
&= -\int_\Omega \sigma_n \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^i \ \ \partial^2_{i,i}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x - \int_\Omega \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^2 \cdot \nabla \widetilde{w}_n^1 \ \ \partial^2_{2,1}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x \\
&\qquad - \int_\Omega \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \ \ \partial^2_{1,2}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x \\
&= -\int_\Omega \sigma_n \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^i \ \ \partial^2_{i,i}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x - \int_\Omega \sigma_n \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \ \ \partial^2_{1,2}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x \\
&\qquad - \int_\Omega \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \ \ \partial^2_{1,2}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x \\
&= -\int_\Omega \sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^i \ \ \partial^2_{i,i}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x - 2\int_\Omega \sigma_n^s \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \ \ \partial^2_{1,2}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x.
\end{aligned}$$

This leads us to the equality

$$\langle f, \varphi(\widetilde{w}_n)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega \sigma_n \nabla u_n \cdot \nabla\left(\varphi(\widetilde{w}_n)\right)\, \mathrm{d}x = -\int_\Omega \sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \ \ \partial^2_{i,j}\varphi(\widetilde{w}_n)\ u_n\, \mathrm{d}x. \tag{3.14}$$

To study the convergence of the last term of (3.14), we first show that $\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j$ is bounded in $L^1(\Omega)$. We have, by periodicity and the Cauchy-Schwarz inequality

$$\begin{aligned}
\int_\Omega |\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j|\, \mathrm{d}x &= \int_\Omega |\Sigma_n^s \nabla \widetilde{W}_n^i \cdot \nabla \widetilde{W}_n^j| \left(\frac{x}{\varepsilon_n}\right)\, \mathrm{d}x \\
&\le C \int_Y |\Sigma_n^s \nabla \widetilde{W}_n^i \cdot \nabla \widetilde{W}_n^j|\, \mathrm{d}x \\
&\le C \sqrt{\int_Y \left|\Sigma_n^s \nabla \widetilde{W}_n^i \cdot \nabla \widetilde{W}_n^i\right| \mathrm{d}x} \sqrt{\int_Y \left|\Sigma_n^s \nabla \widetilde{W}_n^j \cdot \nabla \widetilde{W}_n^j\right| \mathrm{d}x} \\
&\le C \sqrt{(\sigma_n)_* e_i \cdot e_i} \sqrt{(\sigma_n)_* e_j \cdot e_j}
\end{aligned}$$

which is bounded by the hypothesis (3.9). Therefore,

$$\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \text{ is bounded in } L^1(\Omega). \tag{3.15}$$

Due to the periodicity, we know that for $i, j = 1, 2$,

$$2\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j = \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j + \sigma_n^{\mathrm{T}} \nabla \widetilde{w}_n^j \cdot \nabla \widetilde{w}_n^i \rightharpoonup (\sigma_*)^{\mathrm{T}} e_i \cdot e_j + (\sigma_*)^{\mathrm{T}} e_j \cdot e_i = 2\,(\sigma_*)^s e_i \cdot e_j$$

weakly-$*$ in $\mathcal{M}(\Omega)$. Hence, we get that

$$\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \rightharpoonup (\sigma_*)^s e_i \cdot e_j \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{3.16}$$

Moreover, $\partial^2_{i,j}\varphi(\widetilde{w}_n)\ u_n$ converges uniformly to $\partial^2_{i,j}\varphi\ u$. Thus, by passing to the limit in (3.14), we have, again with the Einstein convention

$$\langle f, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\int_\Omega (\sigma_*)^s e_i \cdot e_j \ \ \partial^2_{i,j}\varphi\ u\, \mathrm{d}x = -\int_\Omega \sigma_* : \nabla^2 \varphi\ u\, \mathrm{d}x.$$

Therefore, by integrating by parts and using $\varphi = 0$ on $\partial\Omega$,

$$\int_\Omega \sigma_* \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \langle f, \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}. \tag{3.17}$$

Proof of the corrector result

First of all, we show that the corrector function w_n is bounded in $H^1(\Omega)^2$. By the definition (3.8) of w_n, the Y-periodicity of $W_n^{e_i}$ and the equi-coercivity of Σ_n, we have, for $i = 1, 2$,

$$\alpha \, ||\nabla w_n^i||^2_{L^2(\Omega)^2} \le C\alpha \, ||\nabla W_n^{e_i}||^2_{L^2(Y)^2} \le C \int_Y \Sigma_n \nabla W_n^i \cdot \nabla W_n^i \, \mathrm{d}x = C \, (\sigma_n)_* e_i \cdot e_i \tag{3.18}$$

which is bounded. This inequality combined with the uniform convergence of w_n yields to the boundedness of w_n in $H^1(\Omega)^2$.

Let us consider an approximation $u^\delta \in \mathscr{D}(\Omega)$ of u such that

$$||u - u^\delta||_{H^1_0(\Omega)} \le \delta. \tag{3.19}$$

On the one hand, we have

$$\int_\Omega \sigma_n \nabla u_n \cdot \nabla\big(u_n - u^\delta(w_n)\big) \, \mathrm{d}x = \langle f, \big(u_n - u^\delta(w_n)\big) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

Since w_n converges uniformly to identity on Ω and is bounded in $H^1(\Omega)$ (see (3.18)), with $u^\delta \in \mathscr{D}(\Omega)$, $u^\delta(w_n)$ converges weakly to u^δ in $H^1_0(\Omega)$. Hence, by the weak convergence of u_n to u in $H^1_0(\Omega)$ and (3.19), we can pass to the limit the previous inequality and obtain, for any $\delta > 0$,

$$\limsup_{n\to\infty} \left| \int_\Omega \sigma_n \nabla u_n \cdot \nabla\big(u_n - u^\delta(w_n)\big) \, \mathrm{d}x \right| = \left| \langle f, u - u^\delta \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right| \le C\delta. \tag{3.20}$$

On the other hand, similarly to the proof of the first point (3.14), we are led to the equality

$$\int_\Omega \sigma_n \nabla\big(u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big) \, \mathrm{d}x = -\int_\Omega \sigma_n^s \nabla w_n^i \cdot \nabla w_n^j \;\; \partial^2_{i,j} u^\delta(w_n) \; \big(u_n - u^\delta(w_n)\big) \, \mathrm{d}x. \tag{3.21}$$

As in the first point, $\sigma_n^s \nabla w_n^i \cdot \nabla w_n^j$ is bounded in $L^1(\Omega)$ (see (3.15)), u_n converges uniformly to u and $\partial_{i,j} u^\delta(w_n)$ converges uniformly to $\partial_{i,j} u^\delta$ because u^δ is a $\mathscr{D}(\Omega)$ function. By passing to the limit in (3.21)

$$\int_\Omega \sigma_n \nabla\big(u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big) \, \mathrm{d}x \underset{n\to\infty}{\longrightarrow} -\int_\Omega (\sigma_*)^s e_i \cdot e_j \;\; \partial^2_{i,j} u^\delta \; \big(u - u^\delta\big) \, \mathrm{d}x. \tag{3.22}$$

Moreover, like in (3.17) we have

$$\int_\Omega (\sigma_*)^s e_i \cdot e_j \;\; \partial^2_{i,j} u^\delta \; \big(u - u^\delta\big) \, \mathrm{d}x = \int_\Omega \sigma_* \nabla u^\delta \cdot \nabla\big(u - u^\delta\big) \, \mathrm{d}x. \tag{3.23}$$

By combining this equality with the convergence (3.22), we obtain the inequality

$$\lim_{n\to\infty} \left| \int_\Omega \sigma_n \nabla\big(u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big) \, \mathrm{d}x \right| \le \left| \int_\Omega \sigma_* \nabla u^\delta \cdot \nabla\big(u - u^\delta\big) \right| \tag{3.24}$$

$$\le C|\sigma_*| \; ||\nabla u^\delta||_{L^2(\Omega)^2} ||\nabla\big(u - u^\delta\big)||_{L^2(\Omega)^2} \le C\delta. \tag{3.25}$$

Thus, by adding (3.20) and (3.25), we have

$$\limsup_{n\to\infty} \int_\Omega \sigma_n \nabla\big(u_n - u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big) \, \mathrm{d}x \le C\delta$$

which leads us, by equi-coercivity, to

$$
\begin{aligned}
&\limsup_{n\to\infty}\ \alpha\ ||\nabla(u_n-u^\delta(w_n))||^2_{L^2(\Omega)^2}\\
&\le \limsup_{n\to\infty}\left|\int_\Omega \sigma_n\nabla\big(u_n-u^\delta(w_n)\big)\cdot\nabla\big(u_n-u^\delta(w_n)\big)\ \mathrm{d}x\right|\le C\delta.
\end{aligned}
\tag{3.26}
$$

Thus, the Cauchy-Schwarz inequality, the boundedness of ∇w_n^i in $L^2(\Omega)^2$ (3.18) and the Einstein convention give, for any $\delta>0$,

$$
\begin{aligned}
&||\nabla u_n-\nabla w_n^i\partial_i u||_{L^1(\Omega)^2}\\
&\le ||\nabla u_n-\nabla w_n^i\partial_i u^\delta||_{L^1(\Omega)^2}+||\nabla w_n^i\partial_i\big(u^\delta-u\big)||_{L^1(\Omega)^2}\\
&\le ||\nabla u_n-\nabla w_n^i\partial_i u^\delta||_{L^1(\Omega)^2}+||\nabla w_n^i||_{L^2(\Omega)^2}||\partial_i\big(u^\delta-u\big)||_{L^2(\Omega)}\\
&\le ||\nabla u_n-\nabla w_n^i\partial_i u^\delta||_{L^1(\Omega)^2}+C\delta\\
&\le ||\nabla u_n-\nabla w_n^i\partial_i u^\delta(w_n)||_{L^1(\Omega)^2}+||\nabla w_n^i\big(\partial_i u^\delta-\partial_i u^\delta(w_n)\big)||_{L^1(\Omega)^2}+C\delta\\
&\le ||\nabla u_n-\nabla w_n^i\partial_i u^\delta(w_n)||_{L^1(\Omega)^2}+||\nabla w_n^i||_{L^2(\Omega)^2}||\partial_i u^\delta-\partial_i u^\delta(w_n)||_{L^2(\Omega)}+C\delta\\
&\le ||\nabla u_n-\nabla w_n^i\partial_i u^\delta(w_n)||_{L^1(\Omega)^2}+C||\partial_i u^\delta-\partial_i u^\delta(w_n)||_{L^2(\Omega)}+C\delta.
\end{aligned}
$$

Since $u^\delta\in\mathscr{D}(\Omega)$ and w_n converges uniformly to the identity on Ω, the second term of the last inequality converges to 0. Hence, we get that

$$
\limsup_{n\to\infty}||\nabla u_n-\nabla w_n^i\partial_i u||_{L^1(\Omega)^2}\le\limsup_{n\to\infty}||\nabla u_n-\nabla w_n^i\partial_i u^\delta(w_n)||_{L^1(\Omega)^2}+C\delta.
\tag{3.27}
$$

Finally, this inequality combined with (3.26) gives, for any $\delta>0$,

$$
0\le\limsup_{n\to\infty}||\nabla u_n-\nabla w_n^i\partial_i u||_{L^1(\Omega)^2}\le C\sqrt{\delta}+C\delta,
$$

which implies the corrector result (3.12). □

Remark 3.2 *If the solution u is a $\mathscr{C}^2$ function, then the convergence (3.12) holds true in $L^2_{loc}(\Omega)$ since we may take $u=u^\delta$.*

3.2 A two-phase result

Here, we recall a two-phase result due to G.W. Milton (see [35] pp. 61–65) using the Dykhne transformation.

In order to apply the previous theorem, we reformulate Milton's calculus in such a way that every coefficient depends on n. We then consider, for a fixed n, the periodic homogenization of a conductivity $\sigma_n(h)$ to obtain $(\sigma_n)_*(h)$ through the link between the homogenization of the transformed conductivity and $(\sigma_n)_*(h)$ given by formula (4.16) in [35]. Finally, we study the limit of $(\sigma_n)_*(h)$ through the asymptotic behavior of the coefficients of the transformation, and apply Theorem 3.1 in the example Section 3.3.

In this section we consider a two-phase periodic isotropic medium. Let χ_n be a sequence of characteristic functions of subsets of Y. We define for any $\alpha_1>0$, $\beta_1\in\mathbb{R}$, any sequences $\alpha_{2,n}>0$, $\beta_{2,n}\in\mathbb{R}$ and any $h\in\mathbb{R}$, a parametrized conductivity $\Sigma_n(h)$:

$$
\Sigma_n(h)=(1-\chi_n)(\alpha_1 I_2+h\beta_1 J)+\chi_n(\alpha_{2,n}I_2+h\beta_{2,n}J)\quad\text{in } Y.
\tag{3.28}
$$

We still denote by $\Sigma_n(h)$ the periodic extension to $\mathbb{R}^2$ of $\Sigma_n(h)$ (which satisfies (3.1)). We assume that $\Sigma_n(h)$ satisfies (3.2), and define $\sigma_n(h)$ by (3.3) and $(\sigma_n)_*(h)$ by (3.4).

We have the following result based on an analysis of [35] (pp. 61–65).

Proposition 3.1 *Let χ_n be a sequence of characteristic functions of subsets of Y, $\alpha_1, \alpha_2 > 0$, a positive sequence $\alpha_{2,n}$, $\beta_1, \beta_2 \in \mathbb{R}$, and a sequence $\beta_{2,n}$ such that*

$$\lim_{n\to\infty} \alpha_{2,n} = \infty, \quad \liminf_{n\to\infty} |\beta_{2,n} - \beta_1| > 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta_{2,n}}{\alpha_{2,n}} = \frac{\beta_2}{\alpha_2}. \tag{3.29}$$

Assume that the effective conductivity in the absence of a magnetic field

$$\left(\sigma_n^0\right)_* (\gamma_{1,n}, \gamma_{2,n}) \quad \text{is bounded when} \quad \lim_{n\to\infty} \gamma_{1,n} = \alpha_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\gamma_{2,n}}{\alpha_{2,n}} = \gamma_2 > 0. \tag{3.30}$$

Then, there exist two parametrized positive sequences $\alpha'_{1,n}(h), \alpha'_{2,n}(h)$ such that

$$\lim_{n\to\infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \alpha'_{2,n}(h) \underset{n\to\infty}{\sim} \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}\, \alpha_{2,n}, \tag{3.31}$$

and

$$(\sigma_n)_*(h) = \left(\sigma_n^0\right)_* \left(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\right) + h\beta_1 J + \underset{n\to\infty}{o}(1) \tag{3.32}$$

where $\left(\sigma_n^0\right)_* \left(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\right)$ is bounded.

Remark 3.3 *In view of condition (3.29), the case where $\beta_{2,n}$ tends to β_1 corresponds to perturb the symmetric conductivity*

$$\sigma_n^s = (1-\chi_n)\alpha_1 I_2 + \chi_n \alpha_{2,n} I_2$$

by

$$\sigma_n^s + \beta_1 J + \underset{n\to\infty}{o}(1).$$

Then it is clear that

$$(\sigma_n)_*(h) = \left(\sigma_n^s\right)_* + \beta_1 J + \underset{n\to\infty}{o}(1).$$

Proof of Proposition 3.1. The proof is divided into two parts. After applying Milton's computation (pp. 61–64 of [35]), we study the asymptotic behavior of the different coefficients.

First step: Applying Dykhne's transformation through Milton's computations.

In order to make the Dykhne's transformation following Milton [35] (pp. 62–64), we consider two real coefficients a_n and b_n such that

$$\sigma'_n := \big(a_n \sigma_n(h) + b_n J\big)\big(a_n I_2 + J\sigma_n(h)\big)^{-1} = a_n\big(\sigma_n(h) + (a_n)^{-1} b_n J\big)\big(a_n I_2 + J\sigma_n(h)\big)^{-1} \tag{3.33}$$

is symmetric and, more precisely, according to Notation 1.1, reads as

$$\sigma'_n = (1-\chi_n)\alpha'_{1,n}(h) I_2 + \chi_n \alpha'_{2,n}(h) I_2 = \sigma_n^0\big(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\big). \tag{3.34}$$

Then, using the complex representation

$$\alpha I_2 + \beta J \longleftrightarrow \alpha + \beta i \tag{3.35}$$

suggested by Tartar [41], the constants a_n, b_n must satisfy

$$\alpha'_{1,n}(h) = \frac{a_n(\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \in \mathbb{R} \quad \text{and} \quad \alpha'_{2,n}(h) = \frac{a_n(\alpha_{2,n} + ih\beta_{2,n}) + ib_n}{a_n + i(\alpha_{2,n} + ih\beta_{2,n})} \in \mathbb{R}, \tag{3.36}$$

which implies that

$$b_n = \frac{-a_n^2 h\beta_1 + a_n\Delta_1}{a_n - h\beta_1} = \frac{-a_n^2 h\beta_{2,n} + a_n\Delta_{2,n}}{a_n - h\beta_{2,n}}. \tag{3.37}$$

Denoting $\Delta_1 := \alpha_1^2 + h^2\beta_1^2$ and $\Delta_{2,n} := \alpha_{2,n}^2 + h^2\beta_{2,n}^2$ (thanks to (3.29), n is considered to be larger enough such that $\beta_{2,n} - \beta_1 \neq 0$ and a_n is real), the equality (3.37) provides two non-zero solutions for a_n:

$$a_n = \frac{\Delta_{2,n} - \Delta_1 + \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)}, \tag{3.38}$$

and

$$a_n^- = \frac{\Delta_{2,n} - \Delta_1 - \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)}.$$

The value (3.38) is associated with a positive matrix σ_n', while a_n^- leads us to the negative matrix $\sigma_n^- = -J(\sigma_n')^{-1}J^{-1}$ to exclude (see [34] for more details).

Second step: asymptotic behavior of the coefficients and the homogenized matrix.

One the one hand, by the equality (3.38) combined with (3.29), we have

$$\lim_{n\to\infty} a_n \frac{h(\beta_{2,n} - \beta_1)}{\alpha_{2,n}^2} = \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}$$

which clearly implies that

$$a_n \underset{n\to\infty}{\sim} \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2} \frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)} \quad \text{and} \quad a_n - h\beta_{2,n} \underset{n\to\infty}{\sim} \frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)}. \tag{3.39}$$

On the other hand, (3.29), (3.39) and the first equality of (3.37) give

$$b_n = -a_n h\beta_1 + \Delta_1 + \underset{n\to\infty}{o}(1). \tag{3.40}$$

From (3.29), (3.38), (3.39) and (3.40) we deduce the following asymptotic behavior for the modified phases:

$$\lim_{n\to\infty} \alpha_{1,n}'(h) = \alpha_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha_{2,n}'(h)}{\alpha_{2,n}} = \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}. \tag{3.41}$$

To consider $(\sigma_n')_*$, we need to verify that σ_n' is equi-coercive. We have, by denoting for any $\xi \in \mathbb{R}^2$, $\nu_n = \big(a_n I_2 + J\sigma_n(h)\big)^{-1}\xi$,

$$\forall\ \xi \in \mathbb{R}^2, \quad \sigma_n'\xi\cdot\xi = \big(a_n\sigma_n(h) + b_n J\big)\nu_n \cdot \big(a_n I_2 + J\sigma_n(h)\big)\nu_n = (a_n^2 + b_n)\sigma_n(h)\nu_n\cdot\nu_n$$

and, because $a_n^{-1}\sigma_n(h)$ is bounded in $L^\infty(\Omega)^{2\times 2}$ by (3.39),

$$\forall\ \xi \in \mathbb{R}^2, \quad |\xi| = \big|a_n\nu_n + J\sigma_n(h)\nu_n\big| \leq a_n(1 + C)|\nu_n|.$$

The equi-coercivity of $\sigma_n(h)$ gives

$$\exists\, C > 0, \quad \forall\, \xi \in \mathbb{R}^2, \quad \sigma_n'\xi\cdot\xi \geq \frac{C}{(1+C)^2}\frac{a_n^2 + b_n}{a_n^2}|\xi|^2 \tag{3.42}$$

that is, for n larger enough, by (3.39) and (3.40), σ_n' is equi-coercive.

We can now apply the Keller-Dykhne duality theorem (see, e.g., [30, 23]) to equality (3.33) to obtain

$$(\sigma_n')_* = \big(a_n(\sigma_n)_* + b_n J\big)\big(a_n I_2 + J(\sigma_n)_*\big)^{-1}. \tag{3.43}$$

Moreover, by inverting this transformation, we have

$$(\sigma_n)_*(h) = \big(a_n I_2 - (\sigma_n')_* J\big)^{-1}\big(a_n(\sigma_n')_* - b_n J\big).$$

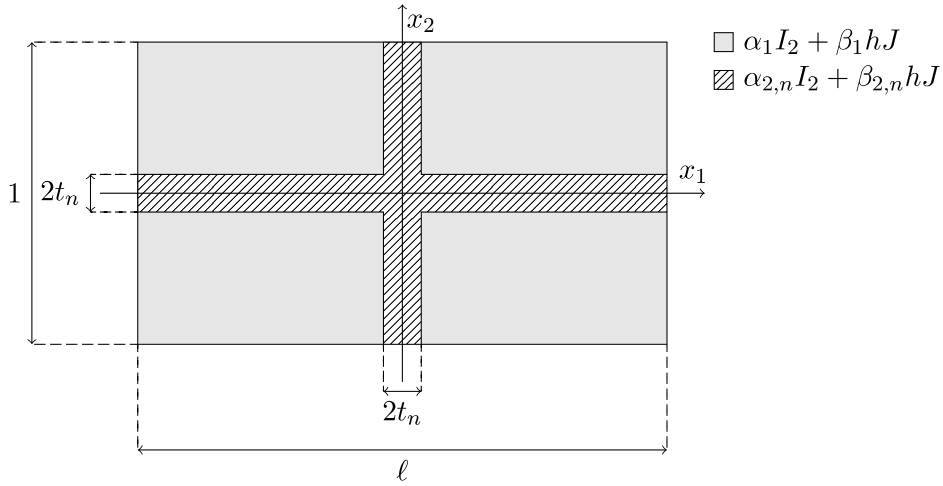


Figure 3.1: The period of the cross-like thin structure

Considering (3.29), (3.39), (3.40), and the boundedness of $(\sigma'_n)_*$ (as a consequence of the bound (3.30)) we get that

$$(\sigma_n)_*(h) = (\sigma'_n)_* - \frac{b_n}{a_n} J + \underset{n\to\infty}{o}(1) = (\sigma'_n)_* + h\beta_1 J + \underset{n\to\infty}{o}(1), \tag{3.44}$$

which concludes the proof taking into account (3.34). $\square$

To derive the limit of $\left(\sigma_n^0\right)_* (\alpha'_{1,n}(h), \alpha'_{2,n}(h))$, we need more information on the geometry of the high conductive phase. To this end, we study the following example.

3.3 A cross-like thin structure

We consider a bounded open subset Ω of $\mathbb{R}^2$ with a Lipschitz boundary, a real sequence ε_n converging to 0, and $f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)$ with $q > 2$. We define, for any $h \in \mathbb{R}$, $\alpha_1, \beta_1 > 0$ and positive sequences $t_n \in (0, 1/2]$, $\alpha_{2,n}$, $\beta_{2,n}$, a parametrized matrix-valued function $\Sigma_n(h)$ from the unit rectangular cell period $Y := (-\frac{\ell}{2}, \frac{\ell}{2}) \times (-\frac{1}{2}, \frac{1}{2})$, with $\ell \geq 1$, to $\mathbb{R}^{2\times 2}$, by (cf. figure 3.1)

$$\Sigma_n(h) := \begin{cases} \alpha_{2,n} I_2 + \beta_{2,n} hJ & \text{in } \omega_n := \{(x_1, x_2) \in Y \mid |x_1|, |x_2| \leq t_n\} \\ \alpha_1 I_2 + \beta_1 hJ & \text{in } Y \setminus \omega_n \end{cases} \tag{3.45}$$

Denoting again by $\Sigma_n(h)$ its periodic extension to $\mathbb{R}^2$, we finally consider the conductivity

$$\sigma_n(h)(x) = \Sigma_n(h)\left(\frac{x}{\varepsilon_n}\right), \quad x \in \Omega, \tag{3.46}$$

and the associated homogenization problem:

$$\mathcal{P}_n \begin{cases} -\mathrm{div}\big(\sigma_n(h)\nabla u_n\big) & = & f & \text{in } \Omega \\ u_n & = & 0 & \text{on } \partial\Omega. \end{cases} \tag{3.47}$$

By virtue of Theorem 3.1 and Proposition 3.1, we focus on the study of the limit of $\left(\sigma_n^0\right)_* \big(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\big)$.

Proposition 3.2 *Let $\sigma_n(h)$ be the conductivity defined by (3.45) and (3.46) and its homogenization problem (3.47). We assume that:*

$$2t_n(\ell+1)\alpha_{2,n} \underset{n\to\infty}{\longrightarrow} \alpha_2 > 0 \quad \text{and} \quad 2t_n(\ell+1)\beta_{2,n} \underset{n\to\infty}{\longrightarrow} \beta_2 > 0. \tag{3.48}$$

Then, the homogenized conductivity is given by

$$\sigma_*(h) = \begin{pmatrix} \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{(\ell+1)\alpha_2} & -h\beta_1 \\ h\beta_1 & \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell+1)\alpha_2} \end{pmatrix}.$$

Remark 3.4 *The previous proposition does not respect exactly the framework defined at the beginning of this section because the period cell is not the unit square $Y = (0,1)^2$: we can nevertheless extend all this section to any type of period cells.*

Remark 3.5 *The condition (3.48) is a condition of boundedness in $L^1(\Omega)^{2\times 2}$ of σ_n because*

$$|\omega_n| = 2t_n(\ell+1) - 4t_n^2 \sim 2t_n(\ell+1),$$

which will ensure the convergence of $(\sigma_n^0)_*$.

Proof of Proposition 3.2. In order to apply Proposition 3.1, we consider two positive sequences $\alpha'_{1,n}(h), \alpha'_{2,n}(h)$ satisfying

$$\lim_{n\to\infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \alpha'_{2,n}(h) \underset{n\to\infty}{\sim} \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}\, \alpha_{2,n}. \tag{3.49}$$

We will study the homogenization of $\sigma'_n := \sigma_n^0\big(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\big)$.

To this end, consider a corrector $W_n^\lambda = \lambda \cdot x - X_n^\lambda$ in the Murat-Tartar sense (see, e.g., [38]) associated with

$$\Sigma'_n := \begin{cases} \alpha'_{2,n}(h)\ I_2 & \text{in } \omega_n = \{(x_1,x_2) \in Y \mid |x_1|, |x_2| \le t_n\} \\ \alpha'_{1,n}(h)\ I_2 & \text{in } Y \setminus \omega_n \end{cases} \tag{3.50}$$

and defined by

$$\begin{cases} \operatorname{div}\big(\Sigma'_n \nabla X_n^\lambda\big) = \operatorname{div}\big(\Sigma'_n \lambda\big) & \text{in } \mathscr{D}'(\mathbb{R}^2) \\ X_n^\lambda \ \text{ is } Y-\text{periodic} \\ \displaystyle\int_Y X_n^\lambda \ \mathrm{d}y = 0. \end{cases} \tag{3.51}$$

On one hand, the extra diagonal coefficients of $(\sigma'_n)_*$ are equal to 0 because, as Σ'_n is an even function on Y, we have, for $i = 1,2$,

$$\begin{cases} y_i \longmapsto W_n^{e_i}(y) & \text{is an odd function,} \\ y_i \longmapsto W_n^{e_j}(y) & \text{is an even function for } i \ne j, \end{cases}$$

which implies that $y_1 \longmapsto \Sigma'_n \nabla W_n^{e_1} \cdot \nabla W_n^{e_2}$ is an odd function. Then, by symmetry of Y with respect to 0,

$$(\sigma'_n)_* e_i \cdot e_j = \int_Y \Sigma'_n \nabla W_n^{e_i} \cdot \nabla W_n^{e_j} \ \mathrm{d}y = 0.$$

On the other hand, as Σ'_n is isotropic, for the diagonal coefficients, we use the Voigt-Reuss inequalities (see, e.g., [29] p.44 or [36]): for any $i = 1,2$ and $j \ne i$,

$$\big\langle \langle (\Sigma'_n e_i \cdot e_i)^{-1} \rangle_i^{-1} \big\rangle_j \le (\sigma'_n)_* e_i \cdot e_i \le \big\langle \langle \Sigma'_n e_i \cdot e_i \rangle_j^{-1} \big\rangle_i^{-1} \tag{3.52}$$

where $\langle\cdot\rangle_i$ denotes the average with respect to y_i at a fixed y_j for $j \ne i$.

An easy computation gives, for the direction e_1,

$$(1-2t_n)\left(\frac{\ell-2t_n}{\ell\alpha'_{1,n}(h)}+\frac{2t_n}{\ell\alpha'_{2,n}(h)}\right)^{-1}+2t_n\left(\frac{\ell}{\ell\alpha'_{2,n}(h)}\right)^{-1}\le(\sigma'_n)_*e_1\cdot e_1$$

and

$$(\sigma'_n)_*e_1\cdot e_1\le\ell\left(\frac{\ell-2t_n}{(1-2t_n)\alpha'_{1,n}(h)+2t_n\alpha'_{2,n}(h)}+\frac{2t_n}{\alpha'_{2,n}(h)}\right)^{-1}.$$

By (3.48) and (3.49), we have the convergence

$$\lim_{n\to\infty}(\sigma'_n)_*e_1\cdot e_1=\alpha_1+\frac{\alpha_2^2+\beta_2^2h^2}{(\ell+1)\alpha_2}.$$

A similar computation on the direction e_2 gives the asymptotic behavior:

$$\lim_{n\to\infty}(\sigma'_n)_*=\lim_{n\to\infty}\left(\sigma_n^0\right)_*\left(\alpha'_{1,n}(h),\alpha'_{2,n}(h)\right)=\begin{pmatrix}\alpha_1+\dfrac{\alpha_2^2+\beta_2^2h^2}{(\ell+1)\alpha_2} & 0\\ 0 & \alpha_1+\dfrac{\alpha_2^2+\beta_2^2h^2}{\ell(\ell+1)\alpha_2}\end{pmatrix}. \tag{3.53}$$

Moreover, the matrix $\sigma_n(h)$ clearly satisfies all the hypothesis of Theorem 3.1. By Theorem 3.1 and (3.53), we have

$$\lim_{n\to\infty}(\sigma_n)_*(h)=\lim_{n\to\infty}\left(\sigma_n^0\right)_*\left(\alpha'_{1,n}(h),\alpha'_{2,n}(h)\right)+\beta_1hJ=\begin{pmatrix}\alpha_1+\dfrac{\alpha_2^2+\beta_2^2h^2}{(\ell+1)\alpha_2} & -h\beta_1\\ h\beta_1 & \alpha_1+\dfrac{\alpha_2^2+\beta_2^2h^2}{\ell(\ell+1)\alpha_2}\end{pmatrix}.$$

We finally apply Theorem 3.1 to get that $\sigma_*(h)=\lim\limits_{n\to\infty}(\sigma_n)_*(h)$. $\square$

4 A three-dimensional fibered microstructure

In this section we study a particular two-phase composite in dimension three. One of the phases is composed by a periodic set of high conductivity fibers embedded in an isotropic medium (figure 4.1a). The conductivity $\sigma_n(h)$ is not symmetric due to the perturbation of a magnetic field.

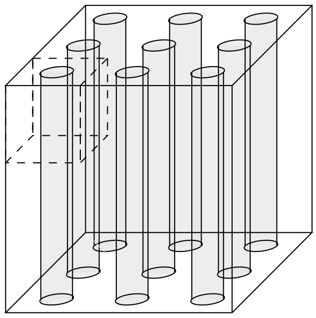
First, describe the geometry of the microstructure. Let $Y:=\left(-\frac{1}{2},\frac{1}{2}\right)^3$ be the unit cube centered at the origin of $\mathbb{R}^3$. For $r_n\in\left(0,\frac{1}{2}\right)$, consider the closed cylinder ω_n parallel to the x_3-axis, of radius r_n and centered in Y:

$$\omega_n:=\left\{y\in Y\mid y_1^2+y_2^2\le r_n^2\right\}. \tag{4.1}$$

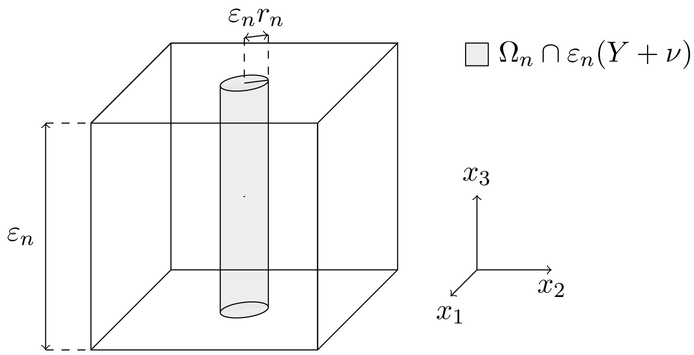
Let $\Omega=\widetilde{\Omega}\times(0,1)$ be an open cylinder of $\mathbb{R}^3$, where $\widetilde{\Omega}$ is a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary. For $\varepsilon_n\in(0,1)$, consider the closed subset Ω_n of Ω defined by the intersection with Ω of the ε_nY-periodic network in $\mathbb{R}^3$ composed by the closed cylinders parallel to the x_3-axis, centered on the points ε_nk, $k\in\mathbb{Z}^2$, in the x_1-x_2 plane, and of radius ε_nr_n, namely:

$$\Omega_n:=\Omega\cap\bigcup_{\nu\in\mathbb{Z}^3}\varepsilon_n(\omega_n+\nu). \tag{4.2}$$

The period cell of the microstructure is represented in figure 4.1b.



(a) The fibers lattice



(b) The period cell

Figure 4.1: The fibered structure in dimension 3

We then define the two-phase conductivity by

$$\sigma_n(h) = \begin{cases} \alpha_1 I_3 + \beta_1 \mathscr{E}(h) & \text{in } \Omega \setminus \Omega_n \\ \alpha_{2,n} I_3 + \beta_{2,n} \mathscr{E}(h) & \text{in } \Omega_n, \end{cases} \tag{4.3}$$

where $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, $\alpha_{2,n} > 0$ and $\beta_{2,n}$ are real sequences, and

$$\mathscr{E}(h) := \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}, \quad \text{for } h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \in \mathbb{R}^3.$$

Our aim is to study the homogenization problem

$$\mathcal{P}_{\Omega,n} \begin{cases} -\mathrm{div}(\sigma_n(h)\nabla u_n) &= f & \text{in } \Omega \\ u_n &= 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Theorem 4.1 *Let $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, and let $\varepsilon_n, r_n, \alpha_{2,n}, \beta_{2,n}$, $n \in \mathbb{N}$, be real sequences such that $\varepsilon_n, r_n > 0$ converge to 0, $\alpha_{2,n} > 0$, and*

$$\lim_{n\to\infty} \varepsilon_n^2 |\ln r_n| = 0, \qquad \lim_{n\to\infty} |\omega_n|\, \alpha_{2,n} = \alpha_2 > 0, \qquad \lim_{n\to\infty} |\omega_n|\, \beta_{2,n} = \beta_2 \in \mathbb{R}. \tag{4.5}$$

Consider, for $h \in \mathbb{R}^3$, the conductivity $\sigma_n(h)$ defined by (4.3).
Then, there exists a subsequence of n, still denoted by n, such that, for any $f \in H^{-1}(\Omega)$ and any $h \in \mathbb{R}^3$, the solution u_n of $\mathcal{P}_{\Omega,n}$ converges weakly in $H_0^1(\Omega)$ to the solution u of

$$\mathcal{P}_{\Omega,*} \begin{cases} -\mathrm{div}(\sigma_*(h)\nabla u_n) &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

where $\sigma_*(h)$ is given by

$$\sigma_*(h) = \alpha_1 I_3 + \left(\frac{\alpha_2^3 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2^2 + \beta_2^2 h_3^2} \right) e_3 \otimes e_3 + \beta_1 \mathscr{E}(h). \tag{4.7}$$

Remark 4.1 *Theorem 4.1 can be actually extended to fibers with a more general cross-section. More precisely, we can replace the disk $r_n D$ of radius r_n by the homothetic $r_n Q$ of any connected open set Q included in the unit disk D, such that the present fiber ω_n is replaced by the new fiber $r_n Q \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ in the period cell of the microstructure.*

On the one hand, this change allows us to use the same test function v_n (4.8) defined in the proof of Theorem 4.1, since v_n remains equal to 1 in the new fibers due to the inclusion $Q \subset D$. On the other hand, Lemma 4.1 allows us to replace the disk D by the open set $Q \subset D$.

Remark 4.2 *We can also extend the result of Theorem 4.1 to an isotropic fibered microstructure composed by three similar periodic fibers lattices arranged in the three orthogonal directions e_1, e_2, e_3, namely*

$$\omega_n := \bigcup_{j=1}^{3} \left\{ y \in Y \mid \sum_{i \neq j} y_i^2 \leq r_n^2 \right\} \quad \text{and} \quad \Omega_n := \Omega \cap \bigcup_{\nu \in \mathbb{Z}^3} \varepsilon_n(\omega_n + \nu),$$

as represented in figure 4.2. Then, we derive the following homogenization conductivity:

$$\sigma_*(h) = \alpha_1 I_3 + \sum_{i=1}^{3} \left(\frac{\alpha_2^3 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2^2 + \beta_2^2 h_i^2} \right) e_i \otimes e_i + \beta_1 \mathscr{E}(h).$$

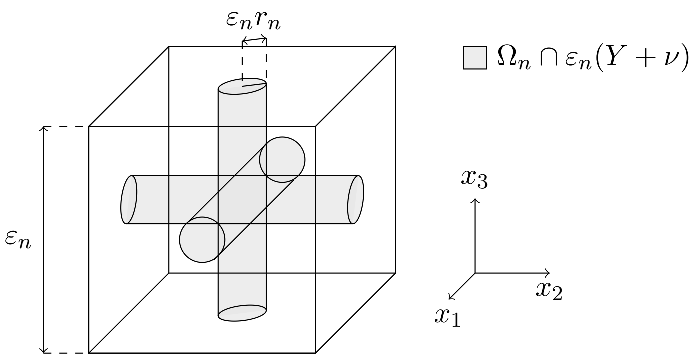


Figure 4.2: The period cell of the isotropic fibered structure in dimension 3

Remark 4.3 *We can check that when the volume fraction $\theta_n = \theta$ and the highly conducting phase of the conductivity $\alpha_{2,n} = \alpha_\theta$ and $\beta_{2,n} = \beta_\theta$ are independent of n, the explicit formula of [27] denoted by $\sigma_*(\theta, h)$, for the classical (since the period cell is now independent of n) periodically homogenized conductivity (see (3.4)) has a limit as $\theta \to 0$ when $\theta\alpha_\theta$ and $\theta\beta_\theta$ converge. Indeed, we may replace in the computations of [27] the optimal Vigdergauz shape by the circular cross-section in the previous asymptotic regime. Therefore, Theorem 4.1 validates the double process characterized by the homogenization at a fixed volume fraction θ combined with the limit as $\theta \to 0$, by one homogenization process in which both the period and the volume fraction $\theta_n = \pi r_n^2$ of the high conductivity phase tend to 0 as $n \to \infty$.*

Remark 4.4 *The hypothesis on the convergence of $\varepsilon_n^2 |\ln r_n|$ (4.5) allows us to avoid nonlocal effects in dimension three (see [24, 1]). These effects do not appear in dimension two as shown in [12]. Therefore, we can make a comparison between dimension two and dimension three based on the strong field perturbation in the absence of nonlocal effects.*

Remark 4.5 *If $h = h_3 e_3$, the homogenized conductivity becomes*

$$\sigma_*(h) = \alpha_1 I_3 + \alpha_2\, e_3 \otimes e_3 + \beta_1 h_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which reduces to the simplified two-dimensional case when the symmetric part of the conductivity is independent of h_3 (i.e. σ_*^0 in (2.40) does not depend on its second argument).

Proof of Theorem 4.1 The proof will be divided into four parts. We first prove the weak-$*$ convergence in $\mathcal{M}(\Omega)$ of $\sigma_n(h)\nabla u_n$ in Ω_n. Then we establish a linear system satisfied by the limits defined by

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i} \rightharpoonup \xi_i \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

Moreover, we deduce from Lemma 4.1 that

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

We finally calculate the homogenized matrix.

We first remark that, classically, the sequence of solutions u_n of $\mathcal{P}_{\Omega,n}$ (see (4.4)) is bounded in $H_0^1(\Omega)$ because, since $\alpha_{2,n}$ diverges to ∞ :

$$||\nabla u_n||^2_{L^2(\Omega)^3} \le C \int_\Omega (\alpha_1 \mathbb{1}_{\Omega\setminus\Omega_n} I_3 + \alpha_{2,n}\mathbb{1}_{\Omega_n} I_3)\nabla u_n \cdot \nabla u_n \, \mathrm{d}x = \int_\Omega \sigma_n(h)\nabla u_n \cdot \nabla u_n \, \mathrm{d}x.$$

By the Poincaré inequality, the previous inequality and (4.4) lead us to

$$||u_n||^2_{H_0^1(\Omega)} \le C||\nabla u_n||^2_{L^2(\Omega)^3} \le C\left|\langle f, u_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right| \le C||f||_{H^{-1}(\Omega)}||u_n||_{H_0^1(\Omega)}$$

and then to

$$||u_n||_{H_0^1(\Omega)} \le C||f||_{H^{-1}(\Omega)}.$$

Thus, up to a subsequence still denoted by n, u_n converges weakly to some function u in $H_0^1(\Omega)$.

First step: Weak-$*$ convergence in $\mathcal{M}(\Omega)$ of the conductivity in the fibers $\mathbb{1}_{\Omega_n}\big(\alpha_{2,n}I_3+\beta_{2,n}\mathscr{E}(h)\big)\nabla u_n$. We proceed as in [22] with a suitable oscillating test function. For $R \in (0, 1/2)$, define the Y-periodic (independent of y_3) function V_n by

$$V_n(y_1,y_2,y_3) = \begin{cases} 1 & \text{if } \sqrt{y_1^2+y_2^2} \le r_n \\ \dfrac{\ln R - \ln\sqrt{y_1^2+y_2^2}}{\ln R - \ln r_n} & \text{if } r_n \le \sqrt{y_1^2+y_2^2} \le R \\ 0 & \text{if } \sqrt{y_1^2+y_2^2} \ge R, \end{cases} \qquad \text{for } y \in Y,$$

and the rescaled function

$$v_n(x) = V_n\left(\frac{x}{\varepsilon_n}\right), \quad \text{for } x \in \mathbb{R}^3. \tag{4.8}$$

In particular, by using the cylindrical coordinates and the fact that r_n converges to 0, this function satisfies the inequalities

$$\begin{aligned} ||v_n||^2_{L^2(\Omega)} \le C||V_n||^2_{L^2(Y)} &= C\left|\ln\frac{R}{r_n}\right|^{-2}\left(\pi r_n^2 + \int_0^{2\pi}\int_{r_n}^R r\ln^2\frac{R}{r}\,\mathrm{d}r\mathrm{d}\theta\right) \\ &= C\left|\ln\frac{R}{r_n}\right|^{-2}\left(\pi\frac{R^2-r_n^2}{2} - \pi r_n^2\ln^2\frac{R}{r_n} - \pi\ln\frac{R}{r_n}\right) \le C\left|\ln\frac{R}{r_n}\right|^{-2}, \end{aligned}$$

$$||\nabla v_n||^2_{L^2(\Omega)^3} \le \frac{C}{\varepsilon_n^2}||\nabla V_n||^2_{L^2(Y)^3} = \frac{C}{\varepsilon_n^2}\left|\ln\frac{R}{r_n}\right|^{-2}\int_0^{2\pi}\int_{r_n}^R \frac{1}{r}\,\mathrm{d}r\mathrm{d}\theta \le \frac{C}{\varepsilon_n^2}\left|\ln\frac{R}{r_n}\right|^{-1}$$

and, consequently

$$||v_n||_{L^2(\Omega)} + \varepsilon_n ||\nabla v_n||_{L^2(\Omega)^3} \leq C \sqrt{\left|\ln \frac{R}{r_n}\right|^{-1}} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.9}$$

Let λ be a vector in $\mathbb{R}^3$ perpendicular to the x_3-axis. Define the Y-periodic function $\widetilde{X}_n$ by $\nabla \widetilde{X}_n = \lambda$ in ω_n, such that $\widetilde{X}_n \in \mathscr{D}(Y)$ and is Y-periodic, and the rescaled function X_n by

$$X_n(x) = \varepsilon_n \, \widetilde{X}_n \left(\frac{x}{\varepsilon_n} \right). \tag{4.10}$$

In particular, X_n satisfies

$$||X_n||_\infty = \varepsilon_n \, ||\widetilde{X}_n||_\infty \leq C\varepsilon_n \quad , \quad ||\nabla X_n||_\infty = ||\nabla \widetilde{X}_n||_\infty \leq C \quad \text{and} \quad \nabla X_n = \lambda \quad \text{in } \Omega_n. \tag{4.11}$$

We have, by (4.11) and (4.9),

$$\begin{aligned} ||v_n X_n||_{H^1(\Omega)} &\leq ||X_n||_\infty ||v_n||_{L^2(\Omega)} + ||X_n||_\infty ||\nabla v_n||_{L^2(\Omega)^3} + ||\nabla X_n||_\infty ||v_n||_{L^2(\Omega)} \\ &\leq C\big(||v_n||_{L^2(\Omega)} + \varepsilon_n ||\nabla v_n||_{L^2(\Omega)^3}\big) \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

which gives

$$\forall \varphi \in \mathscr{D}(\Omega), \quad \varphi \; v_n X_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{strongly in } H^1_0(\Omega). \tag{4.12}$$

Let $\varphi \in \mathscr{D}(\Omega)$. By the strong convergence (4.12), we have

$$\int_\Omega \sigma_n(h) \nabla u_n \cdot \nabla\big(\varphi \; v_n X_n\big) \, \mathrm{d}x = \langle f, \varphi \; v_n X_n \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.13}$$

Let us decompose this integral which converges to 0, into the integral on the fibers set Ω_n and the integral on its complementary:

$$\int_\Omega \sigma_n(h) \nabla u_n \cdot \nabla\big(\varphi \; v_n X_n\big) \, \mathrm{d}x = \int_{\Omega \setminus \Omega_n} (\alpha_1 I_3 + \beta_1 \mathscr{E}(h)) \nabla u_n \cdot \nabla\big(\varphi \; v_n X_n\big) \, \mathrm{d}x \tag{4.14a}$$

$$+ \int_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \mathscr{E}(h)) \nabla u_n \cdot \nabla\big(\varphi \; v_n X_n\big) \, \mathrm{d}x. \tag{4.14b}$$

The expression (4.14a) converges to 0 since, by the Cauchy-Schwarz inequality, the boundedness of u_n in $H^1_0(\Omega)$ and (4.12), we have

$$\left| \int_{\Omega \setminus \Omega_n} (\alpha_1 I_3 + \beta_1 \mathscr{E}(h)) \nabla u_n \cdot \nabla\big(\varphi \; v_n X_n\big) \, \mathrm{d}x \right| \leq |\alpha_1 I_3 + \beta_1 \mathscr{E}(h)| \, ||\nabla u_n||_{L^2(\Omega)^3} \; ||\varphi \; v_n X_n||_{H^1_0(\Omega)} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.15}$$

Consequently, as $v_n = 1$ and $\nabla X_n = \lambda$ on Ω_n, by (4.13), (4.14a), (4.14b) and (4.15), we have

$$\int_{\Omega_n} \sigma_n(h) \nabla u_n \cdot \lambda \; \varphi \, \mathrm{d}x + \int_{\Omega_n} \sigma_n(h) \nabla u_n \cdot \nabla \varphi \; X_n \, \mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0. \tag{4.16}$$

To prove the convergence to 0 of the right term, we now show that $\mathbb{1}_{\Omega_n}\big(\alpha_{2,n} I_3 + \beta_{2,n} \mathscr{E}(h)\big) \nabla u_n$ is bounded in $L^1(\Omega)^3$. We have, by the Cauchy-Schwarz inequality, (4.5) and the classical equivalent $|\Omega_n| \underset{n\to\infty}{\sim} |\Omega| \, |\omega_n|$,

$$\begin{aligned} \left(\int_{\Omega_n} \left| \big(\alpha_{2,n} I_3 + \beta_{2,n} \mathscr{E}(h)\big) \nabla u_n \right| \, \mathrm{d}x \right)^2 &\leq \left| I_3 + \alpha_{2,n}^{-1} \beta_{2,n} \mathscr{E}(h) \right|^2 |\Omega_n| \, \alpha_{2,n} \int_{\Omega_n} \alpha_{2,n} |\nabla u_n|^2 \, \mathrm{d}x \\ &\leq C \int_\Omega \sigma_n(h) \nabla u_n \cdot \nabla u_n \, \mathrm{d}x \\ &\leq C \, ||f||_{H^{-1}(\Omega)} \, ||u_n||_{H^1_0(\Omega)}. \end{aligned}$$

This combined with the boundedness of u_n in $H^1_0(\Omega)$ implies that $\mathbb{1}_{\Omega_n}\big(\alpha_{2,n}I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n$ is bounded in $L^1(\Omega)^3$. This bound and the uniform convergence to 0 of X_n (see (4.11)) imply the convergence to 0 of the right term of (4.16), hence

$$\int_{\Omega_n} \big(\alpha_{2,n}I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n \cdot \lambda\, \varphi\ \mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0.$$

We rewrite this condition as

$$\forall\, \lambda \perp e_3,\quad \mathbb{1}_{\Omega_n}\ \big(\alpha_{2,n}I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n \cdot \lambda \rightharpoonup 0 \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{4.17}$$

Second step: Linear relations between weak-$*$ limits of $\dfrac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\dfrac{\partial u_n}{\partial x_i}$.

Thanks to the Cauchy-Schwarz inequality, we have

$$\left|\left|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i}\right|\right|_{L^1(\Omega)} \le \frac{1}{|\omega_n|}\int_{\Omega_n} |\nabla u_n|\ \mathrm{d}x \le \frac{1}{\sqrt{\alpha_{2,n}|\omega_n|}}\sqrt{\frac{|\Omega_n|}{|\omega_n|}}\sqrt{\int_{\Omega_n}\alpha_{2,n}|\nabla u_n|^2\ \mathrm{d}x}$$

which leads us, by (4.5) and the asymptotic behavior $|\Omega_n| \underset{n\to\infty}{\sim} |\Omega|\ |\omega_n|$, to

$$\left|\left|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i}\right|\right|_{L^1(\Omega)} \le \frac{C}{\sqrt{\alpha_{2,n}|\omega_n|}}\int_{\Omega}\sigma_n(h)\nabla u_n\cdot\nabla u_n\ \mathrm{d}x \le C\left|\langle f, u_n\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\right|$$

which is bounded by the boundedness of u_n in $H^1_0(\Omega)$. This allows us to define, up to a subsequence, the following limits

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i} \rightharpoonup \xi_i \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \quad \text{for } i = 1,2,3. \tag{4.18}$$

Then, by (4.17) we have

$$\big(\alpha_{2,n}I_3 + \beta_{2,n}\mathscr{E}(h)\big)\mathbb{1}_{\Omega_n}\nabla u_n\cdot\lambda = \big(\alpha_{2,n}|\omega_n|I_3 + \beta_{2,n}|\omega_n|\mathscr{E}(h)\big)\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\nabla u_n\cdot\lambda \rightharpoonup 0 \ \text{ weakly-}* \text{ in } \mathcal{M}(\Omega).$$

Therefore, putting $\lambda = e_1, e_2$ in this limit and using condition (4.5), we obtain the linear system

$$\begin{cases} \alpha_2\xi_1 + \beta_2 h_2\xi_3 - \beta_2 h_3\xi_2 = 0 \\ \alpha_2\xi_2 + \beta_2 h_3\xi_1 - \beta_2 h_1\xi_3 = 0 \end{cases} \quad \text{in } \mathcal{M}(\Omega),$$

which is equivalent to

$$\begin{cases} \xi_1 = \dfrac{\beta_2^2 h_1 h_3 - \alpha_2\beta_2 h_2}{\alpha_2^2 + \beta_2^2 h_3^2}\xi_3 \\[2ex] \xi_2 = \dfrac{\beta_2^2 h_2 h_3 + \alpha_2\beta_2 h_1}{\alpha_2^2 + \beta_2^2 h_3^2}\xi_3 \end{cases} \quad \text{in } \mathcal{M}(\Omega). \tag{4.19}$$

Third step: Proof of $\xi_3 = \dfrac{\partial u}{\partial x_3}$.

We need the following result which is an extension of the estimate (3.13) of [21]. The statement of this lemma is more general than necessary for our purpose but is linked to Remark 4.1.

Lemma 4.1 *Let Q be a non-empty connected open subset of the unit disk D. Then, there exists a constant $C > 0$ such that any function $U \in H^1(Y)$ satisfies the estimate*

$$\left|\frac{1}{|r_nQ|}\int_{r_nQ\times\left(-\frac12,\frac12\right)} U\ \mathrm{d}y - \int_Y U\ \mathrm{d}y\right| \le C\sqrt{|\ln r_n|}\ ||\nabla U||_{L^2(Y)^3}. \tag{4.20}$$

Proof of Lemma 4.1. Let $U \in H^1(Y)$. To prove Lemma 4.1, we compare the average value of U on r_nQ and r_nD. Denoting $\widetilde{y} = (y_1, y_2)$, we have, for any $y_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$,

$$\begin{aligned}\left| \fint_{r_nQ} U(\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} - \fint_{r_nD} U(\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} \right| &= \left| \fint_{Q} U(r_n\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} - \fint_{D} U(r_n\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} \right| \\ &\leq \fint_{Q} \left| U(r_n\widetilde{y}, y_3) - \fint_{D} U(r_n\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} \right| \, \mathrm{d}\widetilde{y},\end{aligned}$$

and, since $Q \subset D$,

$$\begin{aligned}\left| \fint_{r_nQ} U(\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} - \fint_{r_nD} U(\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} \right| &\leq \frac{|D|}{|Q|} \fint_{D} \left| U(r_n\widetilde{y}, y_3) - \fint_{D} U(r_n\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} \right| \, \mathrm{d}\widetilde{y} \\ &\leq C \fint_{D} r_n \left(\left| \frac{\partial U}{\partial x_1} \right| + \left| \frac{\partial U}{\partial x_2} \right| \right) (r_n\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y} \\ &= \frac{C}{\pi r_n} \int_{r_nD} \left(\left| \frac{\partial U}{\partial x_1} \right| + \left| \frac{\partial U}{\partial x_2} \right| \right) (\widetilde{y}, y_3) \, \mathrm{d}\widetilde{y},\end{aligned}$$

the last inequality being a consequence of the Poincaré-Wirtinger inequality. Hence, integrating the previous inequality with respect to $y_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and applying the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\left| \fint_{r_nQ \times (-\frac{1}{2}, \frac{1}{2})} U(y) \, \mathrm{d}y - \fint_{r_nD \times (-\frac{1}{2}, \frac{1}{2})} U(y) \, \mathrm{d}y \right| &\leq \frac{C}{\pi r_n} \int_{r_nD \times (-\frac{1}{2}, \frac{1}{2})} |\nabla U| (y) \, \mathrm{d}y \\ &\leq C \sqrt{\int_{r_nD \times (-\frac{1}{2}, \frac{1}{2})} |\nabla U|^2 (y) \, \mathrm{d}y} \\ &\leq C ||\nabla U||_{L^2(Y)^3}.\end{aligned}$$

This combined with the estimate (3.13) of [21], i.e. (4.20) for $Q = D$, and the fact that $\sqrt{|\ln r_n|}$ diverges to ∞ give the thesis. $\square$

Let $\varphi \in \mathscr{D}(\Omega)$. A rescaling of (4.20) with $Q = D$ implies the inequality

$$\left| \frac{1}{|\omega_n|} \int_{\Omega_n} u_n \varphi \, \mathrm{d}x - \int_{\Omega} u_n \varphi \, \mathrm{d}x \right| \leq C \varepsilon_n \sqrt{|\ln r_n|} \, ||\nabla (u_n \varphi)||_{L^2(\Omega)^3} \, .$$

Combining this estimate and the first condition of (4.5) with

$$||\nabla (u_n \varphi)||_{L^2(\Omega)^3} \leq ||\nabla u_n||_{L^2(\Omega)^3} \, ||\varphi||_\infty + ||u_n||_{L^2(\Omega)} \, ||\nabla \varphi||_\infty \leq C,$$

it follows that

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} u_n - u_n \rightharpoonup 0 \;\; \text{in } \mathscr{D}'(\Omega).$$

This convergence does not hold true when $\varepsilon_n^2 |\ln r_n|$ converges to some positive constant. Under this critical regime, non-local effects appear (see Remark 4.4).

Finally, as $\mathbb{1}_{\Omega_n}$ does not depend on the x_3 variable, we have

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} \frac{\partial u_n}{\partial x_3} = \frac{\partial}{\partial x_3} \frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} u_n = \frac{\partial}{\partial x_3} \left(\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} u_n - u_n \right) + \frac{\partial u_n}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} = \xi_3 \;\; \text{in } \mathscr{D}'(\Omega).$$

Fourth step: Derivation of the homogenized matrix.

We now study the limit of $\sigma_n(h)\nabla u_n$ in order to obtain $\sigma_*(h)$. We have

$$\begin{aligned}\sigma_n(h)\nabla u_n\cdot e_1 &= \mathbb{1}_{\Omega\setminus\Omega_n}\left(\alpha_1\frac{\partial u_n}{\partial x_1}-\beta_1 h_3\frac{\partial u_n}{\partial x_2}+\beta_1 h_2\frac{\partial u_n}{\partial x_3}\right)\\ &\qquad +\alpha_{2,n}|\omega_n|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_1}-\beta_{2,n}h_3|\omega_n|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_2}+\beta_{2,n}h_2|\omega_n|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_3}.\end{aligned}\tag{4.21}$$

Hence, passing to the weak-$*$ limit in $\mathcal{M}(\Omega)$ this equality and using the linear system (4.19), $\sigma_n(h)\nabla u_n\cdot e_1$ weakly-$*$ converges in $\mathcal{M}(\Omega)$ to

$$\begin{aligned}&\left(\alpha_1\frac{\partial u}{\partial x_1}-\beta_1 h_3\frac{\partial u}{\partial x_2}+\beta_1 h_2\frac{\partial u}{\partial x_3}\right)+\alpha_2\xi_1-\beta_2 h_3\xi_2+\beta_2 h_2\xi_3\\ &=\big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_1+\alpha_2\frac{\beta_2^2h_1h_3-\alpha_2\beta_2h_2}{\alpha_2^2+\beta_2^2h_3^2}\xi_3-\beta_2h_3\frac{\beta_2^2h_2h_3+\alpha_2\beta_2h_1}{\alpha_2^2+\beta_2^2h_3^2}\xi_3+\beta_2h_2\xi_3\\ &=\big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_1\\ &\quad+\underbrace{\frac{\alpha_2(\beta_2^2h_1h_3-\alpha_2\beta_2h_2)-\beta_2h_3(\beta_2^2h_2h_3+\alpha_2\beta_2h_1)+\beta_2h_2(\alpha_2^2+\beta_2^2h_3^2)}{\alpha_2^2+\beta_2^2h_3^2}}_{=0}\xi_3,\end{aligned}$$

that is

$$\sigma_n(h)\nabla u_n\cdot e_1 \rightharpoonup \big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_1 \quad \text{weakly-}*\text{ in } \mathcal{M}(\Omega).\tag{4.22}$$

The same calculus leads us to

$$\sigma_n(h)\nabla u_n\cdot e_2 \rightharpoonup \big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_2 \quad \text{weakly-}*\text{ in } \mathcal{M}(\Omega).\tag{4.23}$$

We have, for the last direction e_3,

$$\sigma_n(h)\nabla u_n\cdot e_3 \rightharpoonup \left(\alpha_1\frac{\partial u}{\partial x_3}-\beta_1 h_2\frac{\partial u}{\partial x_1}+\beta_1 h_1\frac{\partial u}{\partial x_2}\right)+\alpha_2\xi_3+\beta_2h_2\xi_1-\beta_2h_1\xi_2 \ \text{ weakly-}*\text{ in } \mathcal{M}(\Omega).$$

Hence, again with the linear system (4.19),

$$\begin{aligned}&\left(\alpha_1\frac{\partial u}{\partial x_3}-\beta_1 h_2\frac{\partial u}{\partial x_1}+\beta_1 h_1\frac{\partial u}{\partial x_2}\right)+\alpha_2\xi_3-\beta_2h_2\xi_1+\beta_2h_1\xi_2\\ &=\big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_3+\alpha_2\xi_3-\beta_2h_2\frac{\beta_2^2h_1h_3-\alpha_2\beta_2h_2}{\alpha_2^2+\beta_2^2h_3^2}\xi_3+\beta_2h_1\frac{\beta_2^2h_2h_3+\alpha_2\beta_2h_1}{\alpha_2^2+\beta_2^2h_3^2}\xi_3.\end{aligned}$$

Finally, by the previous equality, (4.22) and (4.23), we get that

$$\sigma_*(h)=\alpha_1 I_3+\left(\frac{\alpha_2^3+\alpha_2\beta_2^2|h|^2}{\alpha_2^2+\beta_2^2h_3^2}\right)e_3\otimes e_3+\beta_1\mathscr{E}(h).$$

$\square$

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